Functions & Relations
Sequences & Series
Limits & Continuity
Lecture 3

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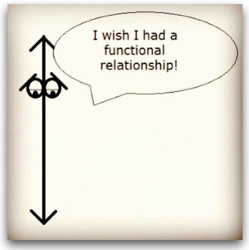
relations and functions

used to compare concepts and uncover relationships between them

- a <u>relation</u> is a relationship between sets of information
- a <u>function</u> is a well-behaved relation



all functions are relations
but not all relations are functions

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relations and functions

example

Consider the set of fruits and the set of colors. We associate fruits with their colors, e.g.,

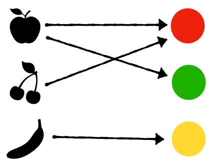
"apple may have the color red"

association

element of first set element of second set

…so the following set of ordered pairs is a **relation**:

{(apple, red), (apple, green), (cherry, red), (banana, yellow)}



...but is this a function?

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relations

- a relation R from the set A to the set B is a subset $A \times B$
  - relation R consists of ordered pairs (a, b) where $a \in A$ and $b \in B$
  - we say ***is related to*** and can write aRb

can be replaced by familiar symbols such as $<, \nless, >, \ngtr, =, \neq$

example

Suppose there are two sets A= {4, 36, 49, 50} and B = {1, -2, -6, -7, 7, 6, 2}

Define *"(a, b) is in the relation R if a is a square of b"*

answer: R ={(4,-2), (4,2),(36,-6), (36,6), (49,-7), (49,7)}

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functions

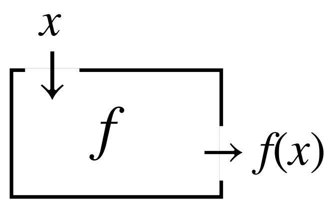
input-output

we define the function f as $f(x): A \to B$ which often is read as as *"f maps A into B"*

we assign this value to a variable y as in $y = f(x)$

what is the rule we follow to obtain $f(x)$?

- computational
- table
- algorithm
- verbal or text
- ⋮



Note: a function must be single valued and can't give back 2 or more results for the same input

So *"$f(16) = 4$ or -4"* is not right!

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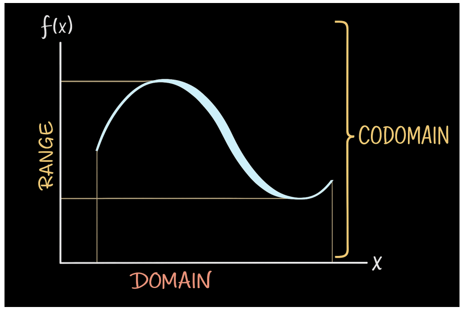
domain, range and codomain

functions $f(x): A \to B$ describe the relationship between two variables as a unique one-to-one mapping where each value of the domain A is mapped to one value of the codomain B



values reached by $x \in A$ are known as image

→ the image is a subset of the codomain B

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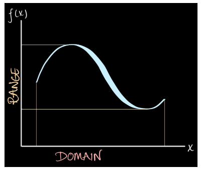
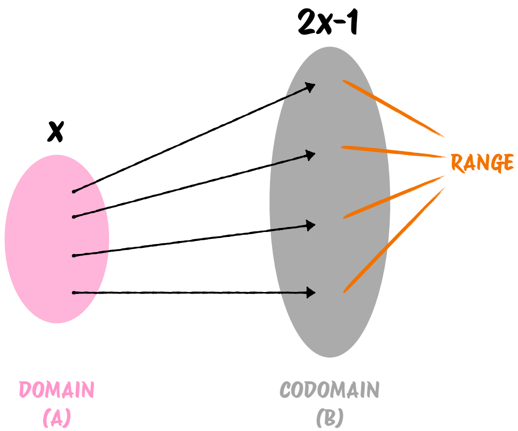
domain, range and codomain





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visual representations of functions and relations

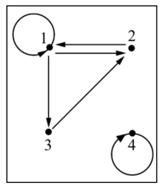
when domain or range of a relation is
- infinite $\Longrightarrow$ not possible to visualize the entire relation
- finite $\Longrightarrow$ lists and graphs can be used

example

let A be a small finite set: $A = \{1,2,3,4\}$

the relation R is defined as $R = \{(1,1), (4,4), (1,3), (3,2), (1,2), (2,1)\}$

graph representation:



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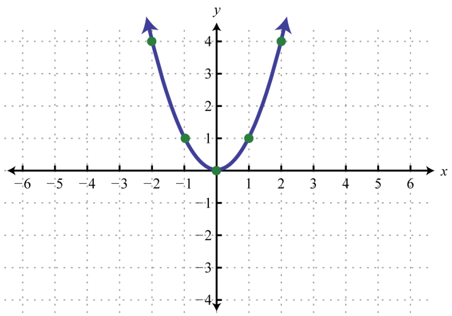
visual representations of functions and relations

example

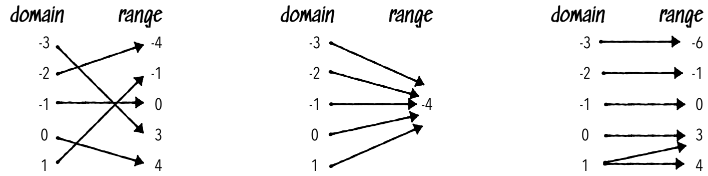
$f(x) = x^2$

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4



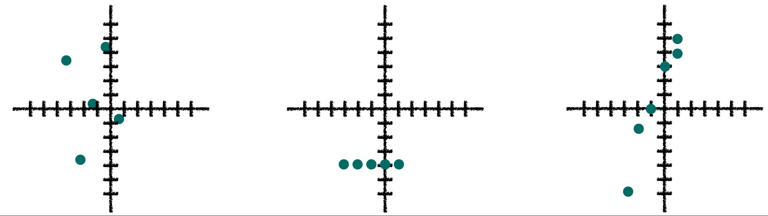
Which points represent the relation $R = \{x, y) \in \mathbb{R} \times \mathbb{R} \mid y \geq x^2\}$?
Is it bounded?

visual representations of functions and relations



are all of these functions?

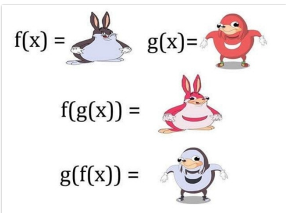
$\Longrightarrow$ *the vertical line test*



function composition

> Let $f : A \rightarrow B$ and $g : C \rightarrow D$. The **composition** of g with f, denoted $g \circ f$, is the function from A to C defined by $g \circ f(x) = g(f(x))$.

- chaining multiple functions: *"g composed with f"*
- order matters!



function composition

example

Consider function composition of the following two functions

$f(x) = 2x + 3$

$g(x) = x^2$

Composition 1: $g \circ f$

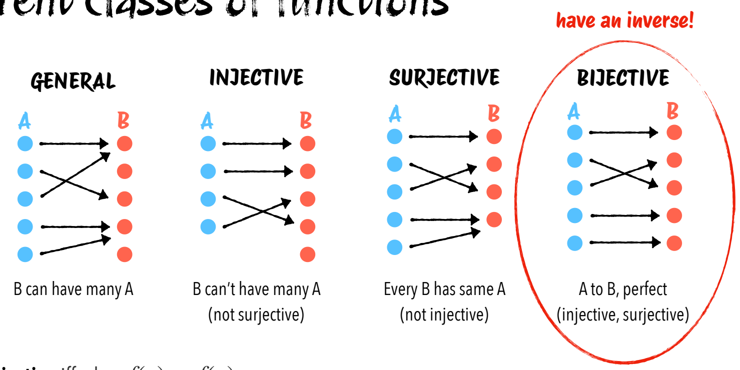
1. First $f(x) = 2x + 3$
2. Then $g(f(x)) = (2x + 3)^2$

Composition 2: $f \circ g$

1. First $g(x) = x^2$
2. Then $f(g(x)) = 2(x^2) + 3 = 2x^2 + 3$

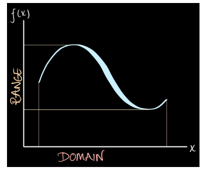
now use specific value $x = 2$

different classes of functions



injective iff when $f(x) = f(y), x = y$
surjective iff $f(A) = B$ or for every y in B, there is at least one x in A such that $f(x) = y$
bijective (from set A to B) if, for every y in B, there is exactly one x in A such that $f(x) = y$

different classes of functions



Let $f : A \to B$ be a function.

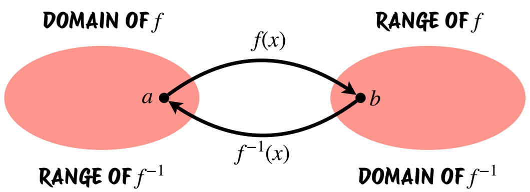
- The function f is said to be **injective** (or **one-to-one**) if for any $x, y \in A$, $f(x) = f(y)$ implies $x = y$.
  Or by contrapositive: $x \neq y$ implies $f(x) \neq f(y)$.
- The function f is said to be **surjective** (or **onto**) if range$(x) = B$.
- If f is both injective and surjective, we say that f is **bijective**.
  - a bijective function is invertible, and so has an inverse.

inverse functions

Suppose $f : A \to B$ is a bijection. Then the inverse of f, denoted

$$f^{-1} : B \to A$$

is the function defined by the rule $f^{-1}(y) = x$ if and only if $f(x) = y$



inverse functions

algorithm

(1) replace $f(x)$ with y in original function

(2) 'switch' instances of x and y (any variables) in original function

(3) solve for y

(4) change y to $f^{-1}(x)$

inverse functions

(1) replace $f(x)$ with y in original function

(2) 'switch' instances of x and y (any variables) in original function

(3) solve for y

(4) change y to $f^{-1}(x)$

example

Let $f(x) = 2x - 3$, then it's inverse is $f^{-1}(x) = \frac{x+3}{2}$.

(1) $y = 2x - 3$

(2) $x = 2y - 3$

(3) $y = \frac{x+3}{2}$

(4) $f^{-1} = \frac{x+3}{2}$

We can check this both ways:

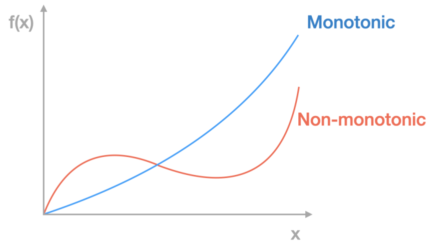
$$f^{-1}(f(x)) = \frac{(2x-3)+3}{2} = \frac{2x}{2} = x$$

$$f(f^{-1}(x)) = 2\left(\frac{x+3}{2}\right) - 3 = x + 3 - 3 = x$$

Since both compositions yield x (the identity function), the functions are indeed inverses

monotonic functions

Monotonicity is the characteristic of order preservation:
it preserves the order of elements from the domain in the range.



"strictly": <

A function is monotonic increasing if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2 \quad \forall x_1$ and $x_2 \in \mathbb{R}$

"strictly" >

A function is monotonic decreasing if $f(x_1) \geq f(x_2)$ whenever $x_1 > x_2 \quad \forall x_1$ and $x_2 \in \mathbb{R}$

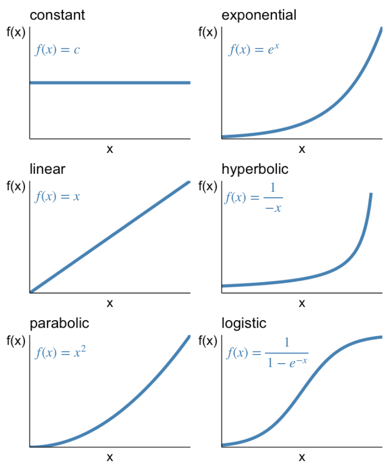
example

$f(x) = 2x + 3$ is monotonically increasing because for any two values x_1 and x_2, then $f(x_1) < f(x_2)$ always.

monotonic functions

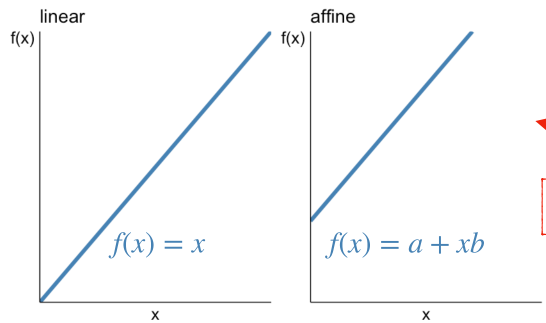
characteristics

- no local extrema
- continuity (when continuous)
- injective



linear function vs. linear equation

needs one point to be determined



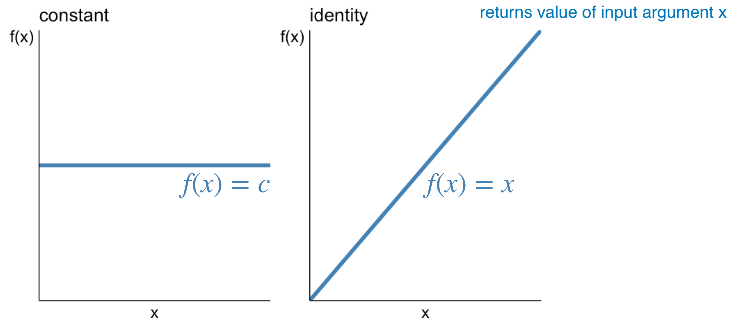
needs two points to be determined

linear functions satisfy

additivity: superposition $f(x_1 + x_2) = f(x_1) + f(x_2)$

scaling: homogeneity $f(ax) = a \cdot f(x) \quad \forall a$

identity function



exponents, roots, logarithms

example: a^n

- How do I solve for x in $a^n = x$? → exponents
- How do I solve for n in $a^n = x$? → logarithms
- How do I solve for a in $a^n = x$? → radicals/roots

sequences

Sequences are an ordered "list" of things $a = \{1,3,5,7,9,\ldots\}$

- can be finite: $\{a_i\}_{i=1}^{n}$ or infinite: $\{a_n\}_{n=1}^{\infty}$
- use curly braces {} and commas as delimiters
- have "rules" that "predict/give" the next value
- values have an order, which identifies them: e.g. a_3 is the third value

Differences between **sequences** and **sets**:

- sets contain every element once, sequences may contain one element many times
- sequences are ordered, whereas order does not matter in sets

example

The following two sequences have patterns:

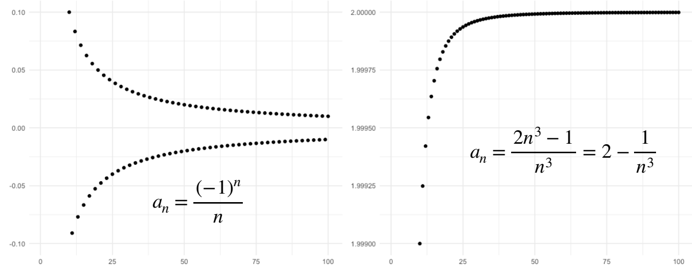
$\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ldots\}$

$\{-1,\ 1,\ -1,\ 1,\ -1,\ldots\}$

sequences

- Sometimes, we can use a function (or algebraic expression) to define the n-th term of a sequence
- A sequence is a function from the positive integers to the real numbers $f : \mathbb{N} \to \mathbb{R}$ with $f(n) = a_n$
- We can draw a graph of this function as a set of points in the plane:

$$(1,a_1),\ (2,a_2),\ (3,a_3),\ \ldots,\ (n,a_n),\ \ldots$$



limit

A limit is the value a function/sequence approaches if argument x approaches some value c

$$\lim_{n \to c} a_n = L$$

or

$$a_n \to L \text{ as } n \to c$$

- $\lim_{n \to c} f(x) = L$ *"the limit of f of x as x approaches c equals L"*
- limits are often challenging to compute, to decide whether a limit exists, we may choose to carry out a convergence test first!
  - convergent $\leftarrow$ finite limit
  - divergent $\leftarrow$ limit DoesNotExist or limit = $\pm\infty$

series

- A series is the summation of a sequence $S = 1 + 3 + 5 + 7 + 9 + \ldots$
  - Finite series you have probably encountered before
  - Infinite series are infinite sums
  - if $a_1 + a_2 + \cdots + a_n = S_n$ then $S_n = \sum_{i=1}^{n} a_i$

examples

$$\sum_{i=1}^{n} a_i$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

geometric series

a geometric series is a series summing the terms of an infinite geometric sequence (the sum of an infinite number of terms), with a constant ratio between them

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| > 1$ the series is divergent.

harmonic series

a harmonic series is the sum of all positive unit fractions

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (does not have a finite limit).

Proof by contradiction.

- Suppose the series converges to S: $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$
- Then: $\frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} + \cdots$
- Therefore, the sum of the odd-numbered terms: $1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} + \cdots$ must be the other half of S
- However this is impossible since $\frac{1}{2n-1} > \frac{1}{2n}$ for each positive integer n.

limit of a function

Let $f(x)$ be a function defined on some open interval that contains a except possibly at a itself. We say that the limit of $f(x)$ as x approaches a is L, and we write:

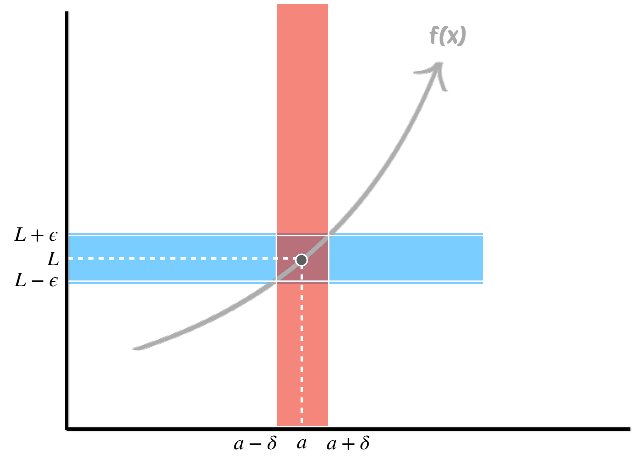
$$\lim_{x\to a} f(x) = L$$

if, for every number $\epsilon > 0$ there exists a number $\delta > 0$ such that whenever

$$0 < |x-a| < \delta, \quad \text{it follows that} \quad |f(x) - L| < \epsilon.$$

- ϵ: this represents how close we want $f(x)$ to be to L. We can choose ϵ to be any small positive number, indicating the "closeness" level we desire
- δ: this represents how close x needs to be to a in order for $f(x)$ to be within ϵ of L

limit of a function



limit of a sequence

How do find the limits? Recall, sequences are really just functions of the integers n…

Theorem

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, where n is an integer, then $\lim_{n\to\infty} a_n = L$.

rules of limits

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is any constant then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \qquad \lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n \qquad \lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c = c \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

limit of series

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ we let s_n denote its n-th partial sum

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

If the sequence s_n is convergent and $\lim_{n\to\infty} s_n = S$ then the series $\sum_{n=1}^{\infty} a_n$ is convergent and we let

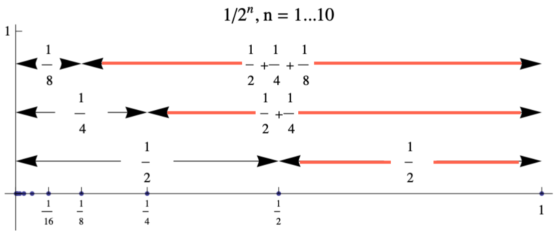
$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_n = \lim_{n\to\infty} s_n = S \longleftarrow \text{sum of the series}$$

Otherwise the series is divergent

determine convergence/divergence using limit of s_n

example

Find the partial sums $s_1, s_2, s_3, \ldots, s_n$ of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Find the sum of the series. Does the series converge?

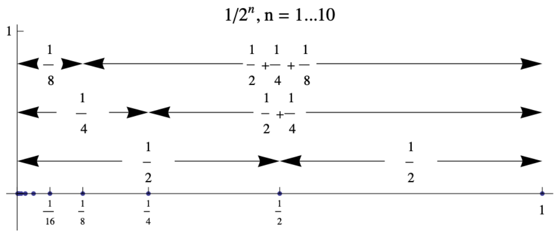


$$s_1 = \frac{1}{2} \qquad s_2 = \frac{1}{2} + \frac{1}{4} \qquad s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \qquad \cdots$$

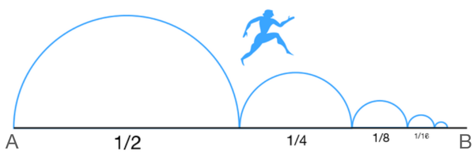
determine convergence/divergence using limit of s_n

example

Find the partial sums $s_1, s_2, s_3, \ldots, s_n$ of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Find the sum of the series. Does the series converge?



Zeno's Paradox



- Consider a runner who is to complete a course from point A to point B.
- Imagine that the runner completes half the distance from A to B,
  and then completes half the remaining distance,
  and again half the remaining distance, and so on…
- Will the runner ever reach point B?

in the end, it all adds up to −1/12

https://www.nytimes.com/2014/02/04/science/in-the-end-it-all-adds-up-to.html

limits of...

- a **sequence** a_i is a number L such that $\lim_{n\to\infty} a_n = L$
- a **series** S_n considers the sum of its elements and is a number S such that $\lim_{n\to\infty} \sum_{i=1}^{n} = S$
- a **function** $y = f(x)$ are values of y given arbitrarily small steps toward an argument $x = c$ such that $\lim_{x\to c} f(x) = L$
  - it is possible to approach the limit from two sides: $\lim_{x\to c^+} f(x) = L^+$ and $\lim_{x\to c^-} f(x) = L^-$
  - the limit exists iff $L^+ = L^- = L$

continuity intuitively

A continuous function's graph does not have sudden breaks

- the pencil test: can you draw the graph without lifting up a pencil?
- the limit test: a function is continuous at argument x, if x exists and is equal to $f(x)$ such that $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = f(c)$

NOTE: a discontinuous function's graph has at least one break in it!

continuity formally

A function $f(x)$ is continuous at a point x_0 if the limit exists at x_0 and is equal to $f(x_0)$

Continuity test:

function is continuous at $f(x)$ if it satisfies the following conditions:

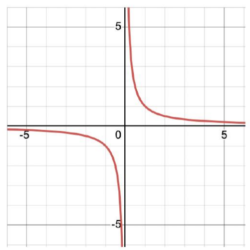
$f(x)$ is defined at c, i.e. $f(c)$ exists

$f(x)$ approaches the same function value to the left and right of c, i.e. $\lim_{x\to c} f(x)$ exists

The function value that $f(x)$ approaches from each side of c is $f(c)$, i.e. $\lim_{x\to c} f(x) = f(c)$

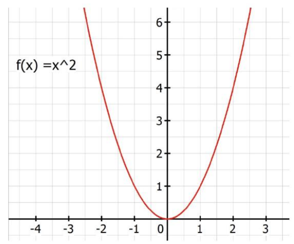
continuity

$f(x) = \frac{1}{x}$



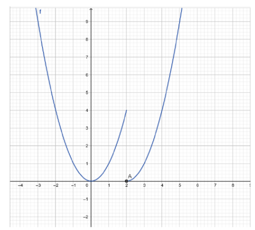
$\lim_{x\to 0} f(x)$?

$f(x) = x^2$



$\lim_{x\to 2} f(x)$?

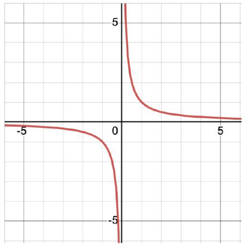
$f(x) = \begin{cases} x^2, & \text{if } x < 2 \\ (x-2)^2, & \text{if } x \geq 2 \end{cases}$



$\lim_{x\to 2} f(x)$?

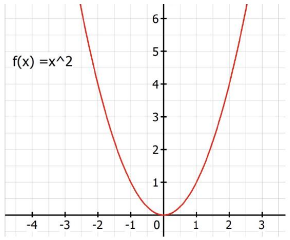
continuity

$f(x) = \frac{1}{x}$



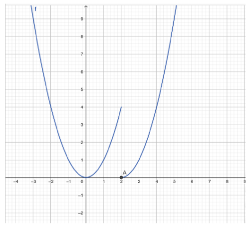
$\lim_{x \to 0} \frac{1}{x} = \frac{1}{0}$ ← DNE

$f(x) = x^2$



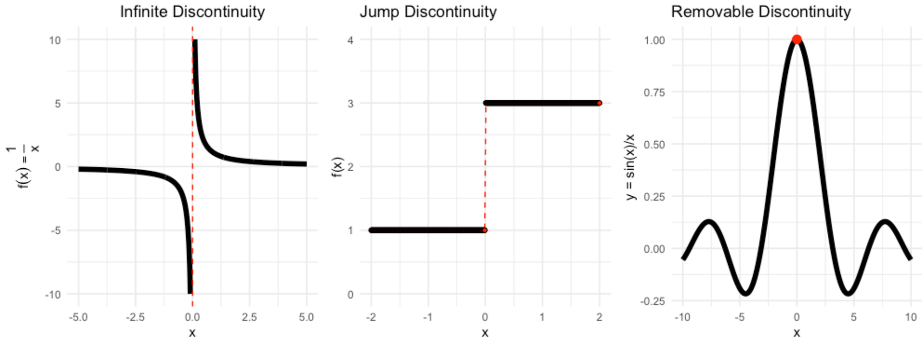
$\lim_{x \to 2} x^2 = 2^2 = 4$

$f(x) = \begin{cases} x^2, & \text{if } x < 2 \\ (x-2)^2, & \text{if } x \geq 2 \end{cases}$



$\lim_{x \to 2^-} x^2 = 2^2 = 4$ and
$\lim_{x \to 2^+} (x-2)^2 = (2-2)^2 = 0^2$
← DNE

discontinuity



plenty more limit and continuity exercises in your tutorial...

continuity and differentiability

Differentiability ⇒ Continuity

but

Continuity ⇏ Differentiability