Calculus Fundamentals: The Integral

Lecture 5

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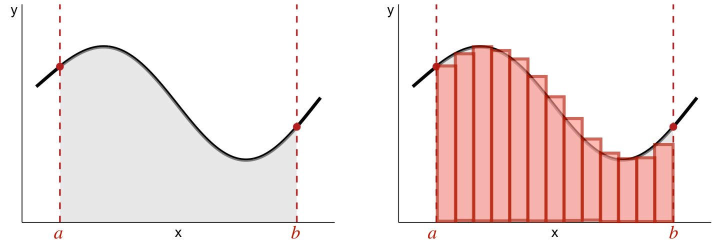
basic idea

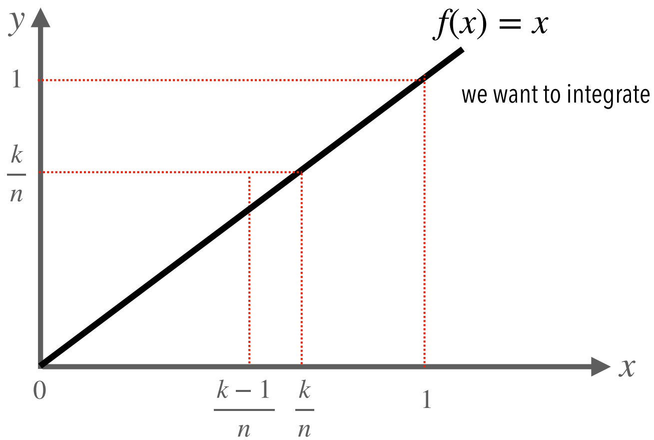
We are interested in calculating areas under curves

1. We divide the interval $a \leq x \leq b$ into pieces (equal length)
2. We build a rectangle on each piece, where the top touches the curve
3. We calculate the total area of the rectangles

We watch what happens as we make the division of the "strips" finer and finer…

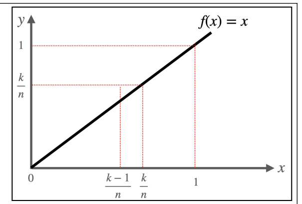


Riemann sum



we want to integrate from $x = 0$ to $x = 1$

Riemann sum



- we divide $[0,1]$ into n equal pieces

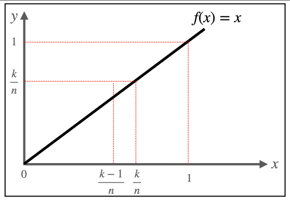
  $\Longrightarrow$ the divisions occur at

$$0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{k-1}{n}, \frac{k}{n}, \cdots, \frac{n-1}{n}, \frac{n}{n} = 1$$

- we have $n+1$ points and we put a rectangle on each point
- the rectangle between $\frac{k-1}{n}$ and $\frac{k}{n}$ has height $f\left(\frac{k}{n}\right) = \frac{k}{n}$ and area of this rectangle is

$$\underbrace{\frac{k}{n}}_{\text{height}} \cdot \underbrace{\frac{1}{n}}_{\text{width}} = \frac{k}{n^2}$$

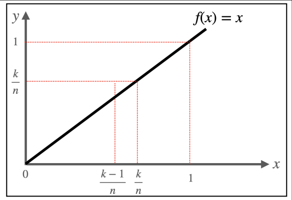
Riemann sum



- The sum of the area of all rectangles on the interval is

$$\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{k}{n^2}+\cdots+\frac{n}{n^2}=\frac{1}{n^2}(1+2+\cdots+k+\cdots+n)$$
$$=\frac{1}{n^2}\frac{n(n+1)}{2}$$
$$=\frac{1}{2}\left(\frac{n+1}{n}\right)$$
$$=\frac{1}{2}\left(1+\frac{1}{n}\right)$$

Riemann sum



- what happens to this expression when $n \to \infty$?

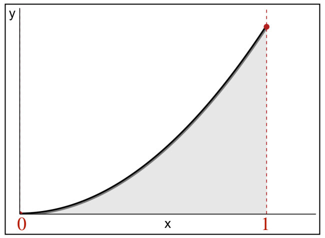
$$\frac{1}{2}\left(1+\frac{1}{n}\right) \to \frac{1}{2} \quad \text{as} \quad n \to \infty$$

- so as we get finer division of rectangles, the sum is approaching the exact area which is $\frac{1}{2}$

Riemann sum

exercise 1

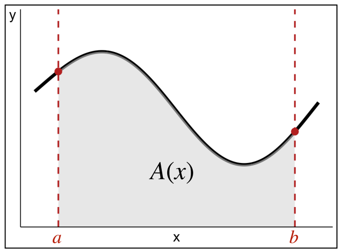
Work out the area bounded by $f(x) = x^2$, the limits $x = 0$ and $x = 1$ and the x-axis (see figure).



we could do this for all functions, but there is a faster way…

area under the curve

- Let's call the curve we want to find the area under $y = f(t)$
- Let the area be a function of x and denote it $A(x)$
- If we know $A(x)$ then we know the are that is $A(b) - A(a)$
- I say $A'(x) = f(x)$, do you believe me? Let's take a look…

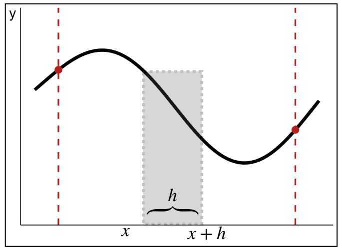


area under the curve

- The difference $A(x+h) - A(x)$ is the area between x and $x+h$
- The area is rectangular (if h is small) with height $f(x)$ and base h so area is $\approx f(x) \cdot h$

$$A(x+h) - A(x) \approx f(x) \cdot h \implies \frac{A(x+h) - A(x)}{h} \approx f(x)$$

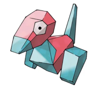
$$\frac{A(x+h) - A(x)}{h} \to f(x) \text{ as } h \to 0$$



By the definition of the derivative, we have $A'(x) = f(x)$ and $A(x)$ as the antiderivative of $f(x)$

note: if $A(x)$ is an antiderivative $f(x)$ then $A(x) + C$ for any constant C is also an antiderivative of $f(x)$

definite and indefinite integral

Σ 

$\int$ 

The indefinite integral of $f(x)$ with respect to x, written as

$$\int f(x)dx = F(x) + c \qquad F'(x) = f(x)$$

The definite integral of $f(x)$ between limits a and b with respect to x, written as

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $f(x)$ is called the integrand

The Fundamental Theorem of Calculus shows that differentiation and integration are inverse processes in two parts based on indefinite and definite integrals

definite and indefinite integral

Indefinite integral

$$\int x^3 - 16x dx = F(x) + C = \frac{x^4}{4} - 8x^2 + C$$

- what function did we differentiate to get $f(x)$?
  → 'backwards' differentiation
- results in an antiderivative
  → always add +C

Definite integral

$$\int_0^4 x^3 - 16x dx = -64$$

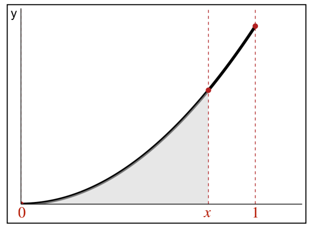
- evaluate integral of $f(x)$ with limits a and b: evaluate antiderivative at upper limit $F(b)$ minus lower limit $F(a)$
- will result in a number

definite and indefinite integral

example

Consider $f(t) = t^2$ and let's calculate the area under $y = t^2$ between 0 and 1.

The area up to x is represented by $A(x) = \int_0^x t^2 dt$] **indefinite integral**



We know $A'(x) = x^2 \implies A(x) = \frac{1}{3}x^3$

This holds for any constant C: $\frac{d}{dx}\left(\frac{1}{3x^3} + C\right) = x^2$

What is our C? $A(0) = \left(\frac{1}{3(0)^3} + C\right) = \int_0^0 t^2 dt = 0 \implies C = 0$

Further we have $A(1) = \frac{1}{3}x^3 \implies \underbrace{\int_0^1 t^2 dt}_{\text{definite integral}} = A(1) - A(0)$ where $A'(x) = x^2$

the fundamental theorem of calculus, part 1

Part 1 is based on indefinite integrals

If $f(x)$ is continuous on an interval, and we define a function:

$$F(x) = \int_a^x f(t)\, dt,$$

then $F'(x) = f(x)$.

- The derivative of the indefinite integral recovers the original function
- Working with indefinite integrals:
  - The answer should be a function + a constant of integration C
  - This expression represents all the possible antiderivatives of $f(x)$

the fundamental theorem of calculus, part 2

Part 2 is based on definite integrals

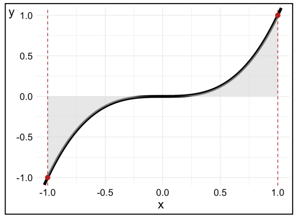
If $F(x)$ is an antiderivative of a continuous function $f(x)$, i.e., $F'(x) = f(x)$, then:

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

- This part uses definite integrals as the primary object and computes them via indefinite integrals
- Working with definite integrals:
  - Find the antiderivative $F(x)$ of $f(x)$
  - We don't have to worry about the constant C here since it cancels out on the RHS
  - The results can be pout in square brackets: $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$
  - Your answer is a number

example

Consider the function $f(x) = x^3$, find the integral over the interval $[-1,1]$



1. Write the integral $\int_{-1}^{1} x^3\, dx$
2. Find the antiderivative $\int x^3\, dx = \frac{x^4}{4} + C$
3. Apply fundamental theorem of calculus

$$\int_{-1}^{1} x^3\, dx = \left[\frac{x^4}{4}\right]_{-1}^{1} \implies \int_{-1}^{1} x^3\, dx = \frac{(1)^4}{4} - \frac{(-1)^4}{4} \implies \int_{-1}^{1} x^3\, dx = \frac{1}{4} - \frac{1}{4} = 0$$

Why Is the integral zero?

The function x^3 is odd, meaning $f(-x) = -f(x)$. For any odd function integrated over a symmetric interval $[-a, a]$ the integral is always zero because the positive and negative contributions cancel out

area under the curve and the value of the definite integral are not always the same

some antiderivatives

function $f(x)$	antiderivative $\int f(x)\, dx$
$f(x) = a$	$\int f(x)\, \mathrm{d}x = ax + C$
$f(x) = ax^n$	$\int f(x)\, \mathrm{d}x = \frac{ax^{(n+1)}}{n+1} + C$
$f(x) = ax^{-1}$	$\int f(x)\, \mathrm{d}x = a \ln\lvert x\rvert + C$
$f(x) = ae^{kx}$	$\int f(x)\, \mathrm{d}x = \frac{1}{k} ae^{kx} + C$
$f(x) = a\cos(kx)$	$\int f(x)\, \mathrm{d}x = \frac{1}{k} a\sin(kx) + C$
$f(x) = a\sin(kx)$	$\int f(x)\, \mathrm{d}x = -\frac{1}{k} a\cos(kx) + C$

rules of bounds

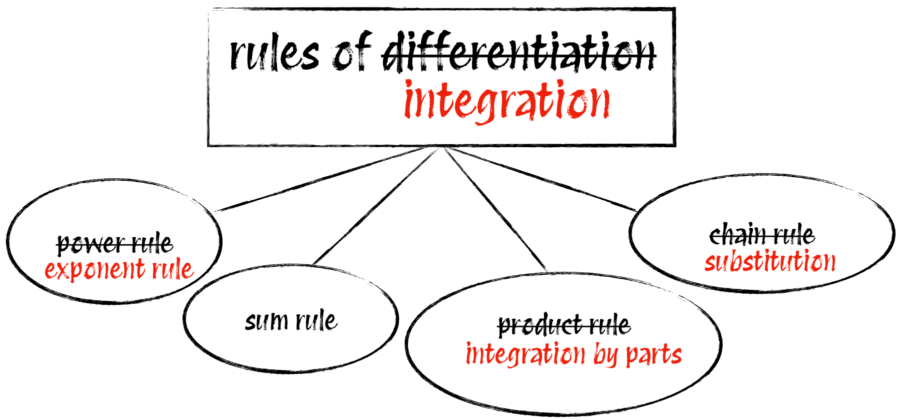
1. Reversing the bounds

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

2. Zero-width interval $\displaystyle\int_a^a f(x)\,dx = 0$

3. Additivity over intervals

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \text{ for any } c \in [a, b]$$



Remember the machine we built for differentiation?
We can use their reverse applications for integration…

the exponent rule

The exponent rule for integration states that for any real $n \neq -1$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

and for the special case when $n = -1$, then

$$\int x^{-1}\,dx = \ln|x| + C\,.$$

example

$$\int x^3\,dx = \frac{x^4}{4} + C$$

(we added 1 to the exponent and divided by the new exponent)

the sum rule

As with differentiation, we also have a sum rule for integration, that is

$$\int \big(f(x) + g(x)\big)\,dx = \int f(x)dx + \int g(x)dx$$

"the antiderivative of a sum is the sum of the antiderivatives"

we also have that $\displaystyle\int Kf(x)dx = K\int f(x)dx$ where K is a constant

example

$$\int (3x^4 + 2x + 5)dx = \int 3x^4 + \int 2xdx + \int 5dx$$

$$= 3\int x^4 + 2\int xdx + 5\int dx$$

$$= \frac{3}{4}x^5 + \frac{2}{3}x^3 + 5x + C$$ (because integral is indefinite)

the exponent and sum rule

exercise 2

Find the indefinite integral $\int (x^2 - 1)(x^4 + 2)dx$

exercise 3

Find the indefinite integral $\int \frac{x^4 + 1}{x^2} dx$

substitution

Substitution is a method to simplify integration by changing variables. For an integral of the form:

$$\int f(g(x))g'(x)\,dx$$

you substitute $u = g(x)$, so $du = g'(x)dx$. This transforms the integral into:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Another way to see it:

$$\int_a^b f(g(u))\,g'(u)\,du = \int_{g(a)}^{g(b)} f(x)\,dx$$

substitution: step by step

$$\int_a^b f(g(u))\,g'(u)\,du = \int_{g(a)}^{g(b)} f(x)\,dx$$

1. Prepare substitution:
   - ➡ find substitute term u (should not be a constant)
   - ➡ solve for x
   - ➡ differentiate $g(u)$
   - ➡ replace integration variable
2. Substitution

   ⚠ ***Warning:*** *by now the integral should be solely in terms of* u.
   If there are still terms containing x *at this stage, stop and consider another choice of* u.
3. Integration (don't forget +C if you are working with indefinite integrals)
4. 'Substitute backwards' (this step only if indefinite integrals)

substitution

$$\int_a^b f(g(u))\,g'(u)\,du = \int_{g(a)}^{g(b)} f(x)\,dx$$

example

Compute the indefinite integral of $f(x) = e^{3x}$.

1. Preparation:
   - ➡ find substitute term u ('inner function') $\implies u = 3x$
   - ➡ solve for x $\implies x = \frac{u}{3} \implies g(u) = \frac{1}{3}u$
   - ➡ differentiate $g(u)$ $\implies g'(u) = \frac{1}{3}$
   - ➡ replace integration variable $\implies dx = g'(u)du = \frac{1}{3}du$

substitution

$\int_a^b f(g(u))\, g'(u)\, du = \int_{g(a)}^{g(b)} f(x)\, dx$

example

Compute the indefinite integral of $f(x) = e^{3x}$.

2. Substitution: $\int e^{3x} dx$ with $u = 3x$ and $dx = \frac{1}{3} du$

$$\implies \int e^u \cdot \frac{1}{3} du = \frac{1}{3} \int e^u du$$

3. Integration

$$F(u) = \frac{1}{3} e^u + C$$

4. 'Substitute' backwards

$$\implies F(x) = \frac{1}{3} e^{3x} + C$$

integration by parts

Suppose we have two function $f(x)$ and $g(x)$ which we write shortly as f and g.
Then **the product rule** states

$$\frac{d}{dx}(fg) = f'g + fg'$$

Rearranging gives

$$fg' = \frac{d}{d(x)}(fg) - f'g$$

Integrating both sides w.r.t. x gives

$$\int fg' dx = \int \frac{d}{d(x)}(fg) - \int f'g dx$$

$$= fg - \int f'g dx$$

integration by parts

We write the rule for integration by parts as

$$\int f(x) \frac{dg(x)}{dx} dx = \underbrace{f(x)g(x)}_{\text{boundary term}} - \underbrace{\int g(x) \frac{dg(x)}{dx} dx}_{\text{integral term}}$$

Another way to see it:

$$\int f(x)\, g'(x)\, dx = f(x)g(x) - \int f'(x)\, g(x)\, dx$$

plugging into the above equation (hopefully) leads to an easier integration

integration by parts: step by step

$\int f(x)\, g'(x)\, dx = f(x)g(x) - \int f'(x)\, g(x)\, dx$

1. Choose $f(x)$ and $g'(x)$
2. Find $f(x) \Longrightarrow f'(x)$ and $g'(x) \Longrightarrow g(x)$
3. Plug your functions into the integration by parts formula
4. Simplify and solve

integration by parts: step by step

$\int f(x)\, g'(x)\, dx = f(x)g(x) - \int f'(x)\, g(x)\, dx$

example

Evaluate the indefinite integral $\int x^3 \ln(x) dx$.

1. Choose $f(x)$ and $g'(x)$

   $f(x) = \ln(x)$ and $g'(x) = x^3$

2. Find $f(x) \Longrightarrow f'(x)$ and $g'(x) \Longrightarrow g(x)$

   $f'(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x^4}{4}$

3. Plug your functions into the integration by parts formula

$$\int f(x)\, g'(x)\, dx = f(x)g(x) - \int f'(x)\, g(x)\, dx = \ln(x) \cdot \left(\frac{x^4}{4}\right) - \int \frac{1}{x} \cdot \left(\frac{x^4}{4}\right) dx$$

integration by parts: step by step

$\int f(x)\, g'(x)\, dx = f(x)g(x) - \int f'(x)\, g(x)\, dx$

example

Evaluate the indefinite integral $\int x^3 \ln(x) dx$.

4. Simplify and solve

$$\begin{aligned}
\int x^3 \ln(x)\, dx &= \ln(x) \cdot \left(\frac{x^4}{4}\right) - \int \frac{1}{x} \cdot \left(\frac{x^4}{4}\right) dx \\
&= \frac{x^4 \ln(x)}{4} - \frac{1}{4} \int x^3\, dx \\
&= \frac{x^4 \ln(x)}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + C \\
&= \frac{x^4 \ln(x)}{4} - \frac{x^4}{16} + C
\end{aligned}$$

when to use which method?

Integration by substitution

- 'chain rule' of integration
- there is an 'outer function', containing an 'inner function'
- the 'inner function' can be differentiated easily

rule of thumb: try substitution first!

Integration by parts

- 'product rule' of integration
- two functions are multiplied with each other
- one function is an 'easy' derivative
- the other can be differentiated

practice makes perfect