Extrema in One Dimension

Lecture 6

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extrema of a function

one of the most important applications of calculus is optimization of functions

This is basically understanding the behavior of a function f on a given interval I

- Does f have a maximum?
- Does f have a minimum?
- Where is the function increasing?
- Where is the function decreasing?

We use derivatives to answer these questions and at the end of today's lecture you'll even see how we can use derivatives to approximate a function…

subclasses of extrema

Extrema can be divided in the following subsections

- Maxima and minima
- Absolute (or global) and local (or relative) extrema

Note: Extrema, Maxima and Minima are the plural form of Extremum, Maximum and Minimum, respectively

example

$f(x) = x^2 + 1$ $\qquad$ $f(x) = x^2 + 1$ $\qquad$ $f(x) = \begin{cases} x^2 + 1, & x \neq 0 \\ 2, & x = 0 \end{cases}$

extrema can occur in interior points or endpoint of an interval,
extrema occurring at endpoints are called endpoint extrema

definition: absolute extrema

Let $f(x)$ be a function defined on interval I (open, closed, or half-open) and let $a \in I$

- We say $f(x)$ has an absolute maximum at $x = a$ if $f(a)$ is the maximal value of $f(x)$ on I:

$$f(a) \geq f(x) \text{ for all } x \in I$$

- We say $f(x)$ has an absolute minimum at $x = a$ if $f(a)$ is the minimal value of $f(x)$ on I:

$$f(a) \leq f(x) \text{ for all } x \in I$$

the extreme value theorem

If f is continuous on a closed interval $[a, b]$ then f has both a minimum and a maximum

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definition: local extrema

Let $f(x)$ be a function.

- We say that $f(x)$ has an local maximum at $x = a$ if $f(a)$ is the maximal value of $f(x)$ on some open interval I inside the domain of f containing a.
- We say that $f(x)$ has an local minimum at $x = a$ if $f(a)$ is the minimal value of $f(x)$ on some open interval I inside the domain of f containing a.

we look for "valleys" and "peaks" of a function

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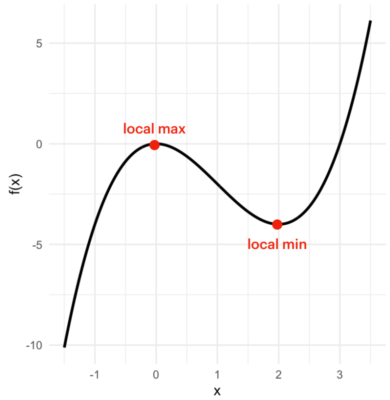
example

$f(x) = x^3 - 3x^2$



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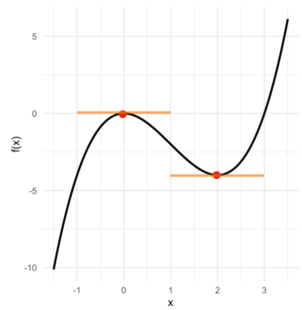
the first derivative test for local extreme values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.



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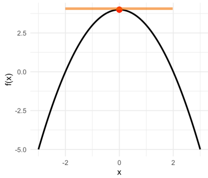
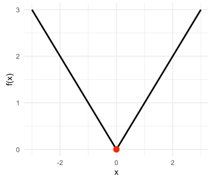
critical point

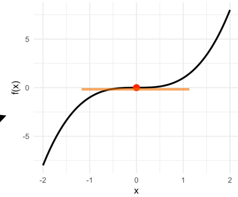
An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.





If $f(c)$ is a local maximum or minimum, then c is a critical point of $f(x)$

Note: the reverse does not hold, i.e., if $f'(c) = 0$ then $f(c) = 0$ is not necessarily a maximum or minimum.



finding the absolute extrema

Suppose that $f(x)$ is continuous on the closed interval $[a, b]$. Then $f(x)$ attains its absolute maximum and minimum values on $[a, b]$ at either:

- a critical point
- one of the end points

finding extrema: step by step

General

1. Find $f'(x)$
2. Set $f'(x_0) = 0$ and solve for all x_0
   $\rightarrow$ stationary points
3. Find $f''(x)$
4. For each stationary point, plug x_0 into $f''(x)$ and determine all extrema, inflection points, and saddle points
5. Plug each point into $f(x)$ to find the corresponding value of y

With given interval $[a, b]$

5. Evaluate the function's values at the lower limit $f(a)$ and the upper limit $f(b)$
6. Compare all extrema and determine the global minimum and maximum on the interval $[a, b]$

Very detailed algorithm in Moore & Siegel, 2013, p.168

finding the absolute extrema

example

Find the absolute maximum and minimum values of $f(x) = 3x - x^3$ on the interval $[-1,3]$.

1. Find $f'(x)$: $f(x) = 3x - x^3 \implies f'(x) = 3 - 3x^2 = 3(1 - x^2)$
2. Set $f'(x_0) = 0$ and solve (stationary points)
   $$3(1 - x^2) = 0 \implies 1 - x^2 = 0 \implies x^2 = 1 \implies x = \pm 1$$
   stationary points: $x_0 = -1,1$ (both within interval)
3. Find $f''(x)$: $f''(x) = \frac{d}{dx}(3 - 3x^2) = -6x$
4. Classify each stationary point with $f''(x_0)$
   $x_0 = 1 : f''(1) = -6(1) = -6 < 0 \Rightarrow$ local maximum
   $x_0 = -1 : f''(-1) = -6(-1) = 6 > 0 \Rightarrow$ local minimum
5. Now find their y-values: $f(1) = 3(1) - 1^3 = 3 - 1 = 2$ and $f(-1) = 3(-1) - (-1)^3 = -2$

finding the absolute extrema

example

Find the absolute maximum and minimum values of $f(x) = 3x - x^3$ on the interval $[-1,3]$.

6. Evaluate f at the endpoints $a = -1,\ b = 3$

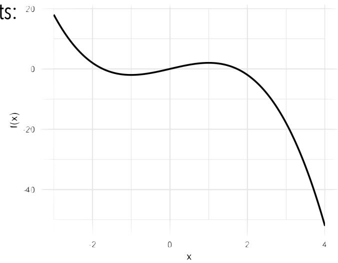
   we already have $f(-1) = -2$
   compute $f(3)$: $f(3) = 3 \cdot 3 - 3^3 = 9 - 27 = -18$

7. Make a table with the critical points inside the interval and its endpoints:

x	$3x - x^3$
1	$3 - 1 = 2$
-1	$-3 - (-1) = -2$
3	$3 \cdot 3 - 3^3 = -18$

Absolute maximum: $f(1) = 2$ at $x = 1$

Absolute minimum: $f(3) = -18$ at $x = 3$



finding the absolute extrema

exercise 1

Find the absolute maximum and minimum value of $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

recall: the second derivative

- To characterize troughs and humps we need the second derivative
- We can view $f'(x)$ itself as a function that we differentiate it again:

$$\frac{d}{dx}\left(f'(x)\right) = \frac{d^2}{dx^2}\left(f(x)\right) = f''(x)$$

The geometric interpretation of f'':

1. $f''(x) > 0$
2. $f''(x) < 0$
3. $f''(c) = 0$

the second derivative test for local extreme values.

The geometric interpretation of f'':

1. If $f''(x) > 0$ then the slope of the tangent line is increasing in value
   $\Longrightarrow$ if $f'(c) = 0$ and $f''(c) > 0$, then around c, $f(x)$ is a trough/valley
   $\Longrightarrow$ we can expect a local minimum value of f at c

2. If $f''(x) < 0$ then the slope of the tangent line is decreasing in value
   $\Longrightarrow$ if $f'(c) = 0$ and $f''(c) < 0$, then around c, $f(x)$ is a hump/peak
   $\Longrightarrow$ we can expect a local maximum value of f at c

3. If $f'(c) = 0$ and $f''(c) = 0$ then the slope often doesn't change sign
   i.e. it goes from positive slope to zero to positive slope (decreasing to zero then increasing), or negative to zero to negative (increasing to zero then decreasing).

saddle or inflection points

Inflection points

- does not have to be stationary point, but if it is, then **not** a local extremum but a saddle point
- sign of curvature of function changes
- "car steering" test: Imagine you're driving a car; do you have to make an S-curve to follow the curvature of the graph?

Saddle points

- stationary point that is not a local extremum
- slope is equal to zero for **all** directions of graph, i.e. tangent is horizontal → there is no sign change before and after saddle point
- conditions:
  - ✓ $f'(x_0) = 0$
  - ✓ $f''(x_0) = 0$
  - ✓ $f'''(x_0) \neq 0$

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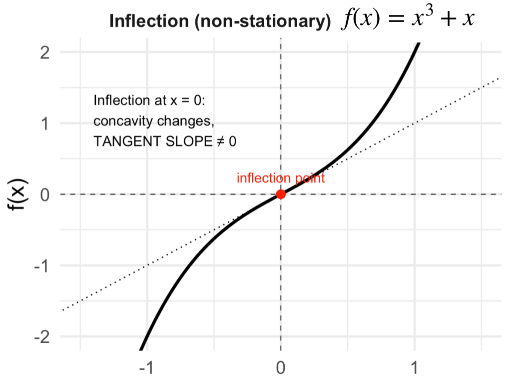
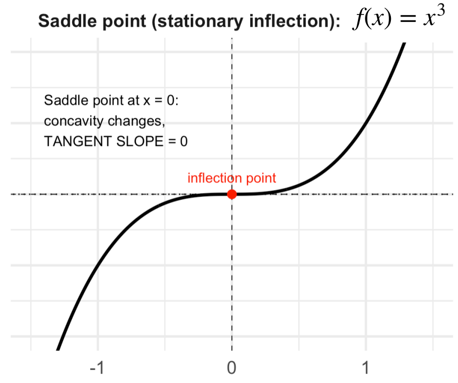
saddle or inflection points





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saddle or inflection points: algorithms

Inflection points

- find $f'(x)$
- find $f''(x)$
- set $f''(x) = 0$ and find inflection points (or points where f is undefined)
- find $f'''(x)$ and check if it is unequal to 0 and changes sign
- plug in $x_0 = a$ into $f'(x)$ and check if $f'(x_0) \neq 0$

Saddle points

- find $f'(x)$
- find $f''(x)$
- set $f''(x) = 0$ and find inflection points (or points where f is undefined)
- find $f'''(x)$ and check if it is unequal to 0 and changes sign
- plug in $x_0 = a$ into $f'(x)$ and check if $f'(x_0) \neq 0$

conditions:
- ✓ $f'(x_0) = 0$
- ✓ $f''(x_0) = 0$
- ✓ $f'''(x_0) \neq 0$

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concavity, convexity and inflection points

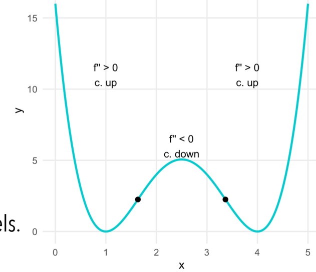
Intuition: an inflection point tells us where the sign of curvature of the function changes

- To describe concavity & convexity, f must be differentiable at least twice on interval $I = [a, b]$
- The graph of f is convex (concave up) if $f'' > 0$ → *the gradient is increasing*
- The graph of f is concave (concave down) if $f'' < 0$ → *the gradient is decreasing*

Example

a quartic $f(x) = (x-1)^2(x-4)^2$ which has
- concave up on the left,
- concave down in the middle,
- concave up on the right,

and marks the inflection points with black dots, where the $f'' > 0$ and $f'' < 0$ are noted with c. up/c. down labels.



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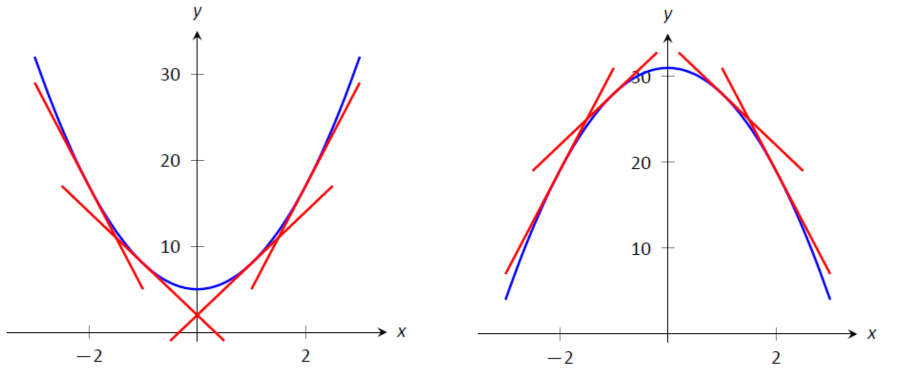
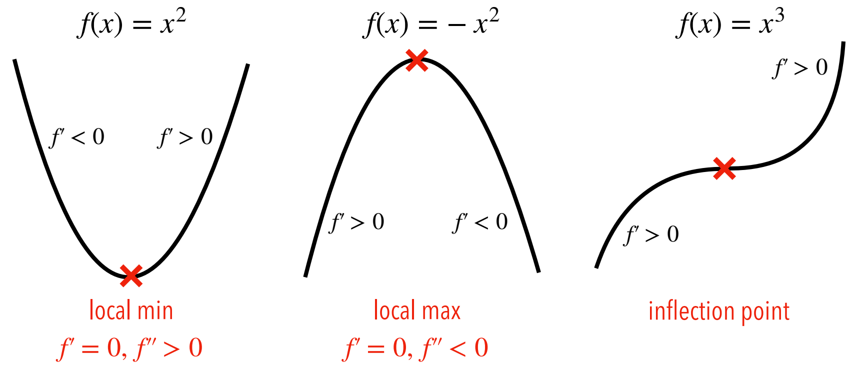
concavity, convexity and inflection points



more examples



hands on exercises in finding these points in your tutorial....

approximating functions

- The first derivative tells us whether the function f is increasing or decreasing.
- The second derivative describes the curvature of f.
- Can we reconstruct a function using information from all its derivatives?

YES!

derivative information at a point → output information near that point

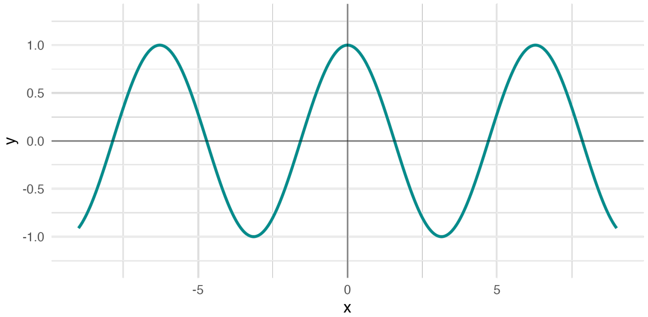
by taking non-polynomial functions and turning them into polynomials

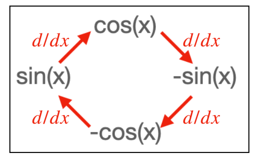
example: cos(x)

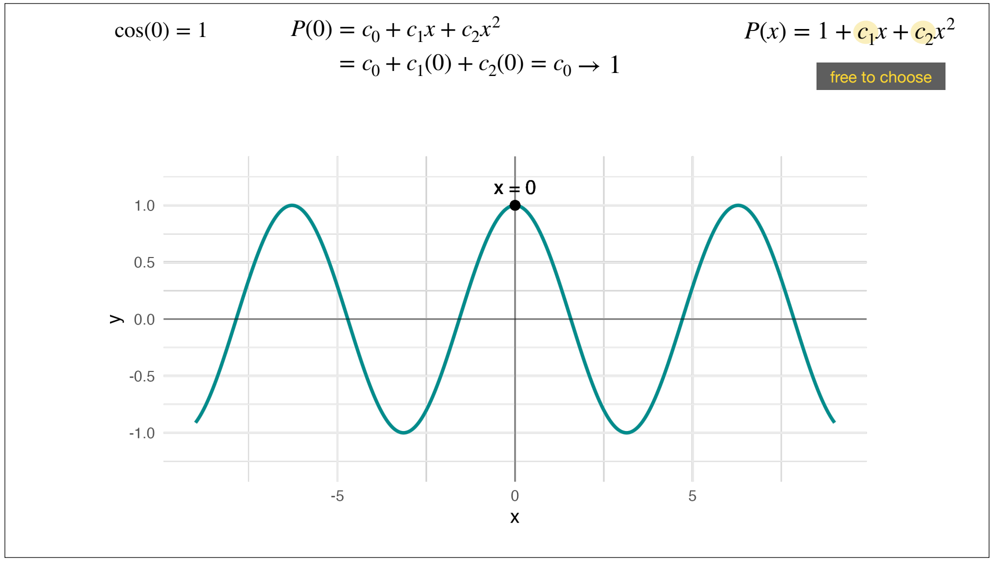
Let's look at approximating $f(x) = \cos(x)$ at point $x = 0$

using polynomial of order two: $P(x) = c_0 + c_1 x + c_2 x^2$

free to choose







$\cos(0) = 1$
$P(0) = c_0 + c_1x + c_2x^2$
$= c_0 + c_1(0) + c_2(0) = c_0 \to 1$
$P(x) = 1 + c_1x + c_2x^2$
free to choose
x = 0
y
x

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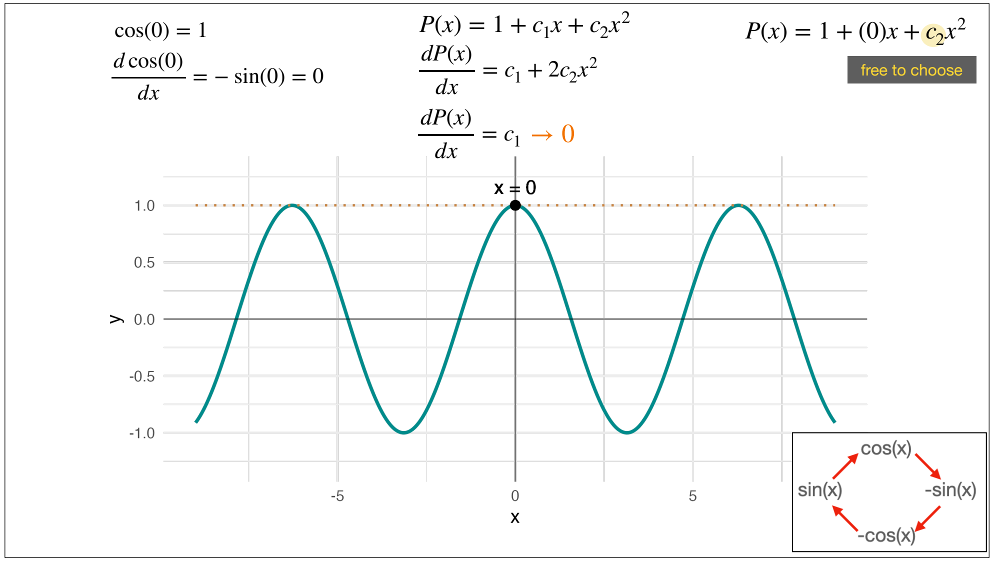

$\cos(0) = 1$
$\frac{d\cos(0)}{dx} = -\sin(0) = 0$
$P(x) = 1 + c_1x + c_2x^2$
$\frac{dP(x)}{dx} = c_1 + 2c_2x^2$
$\frac{dP(x)}{dx} = c_1 \to 0$
$P(x) = 1 + (0)x + c_2x^2$
free to choose
x = 0
y
x
cos(x)
sin(x)
-sin(x)
-cos(x)

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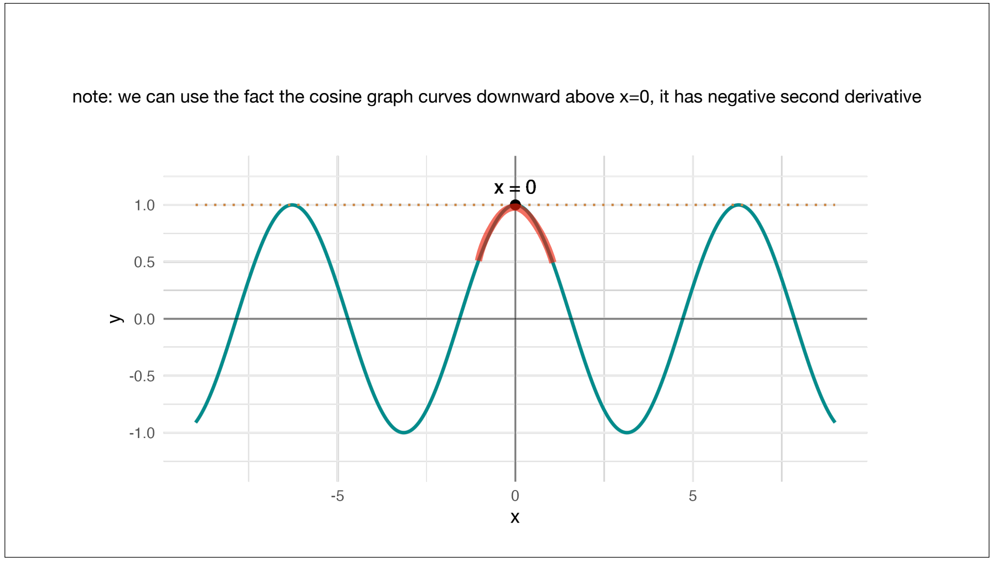

note: we can use the fact the cosine graph curves downward above x=0, it has negative second derivative
x = 0
y
x

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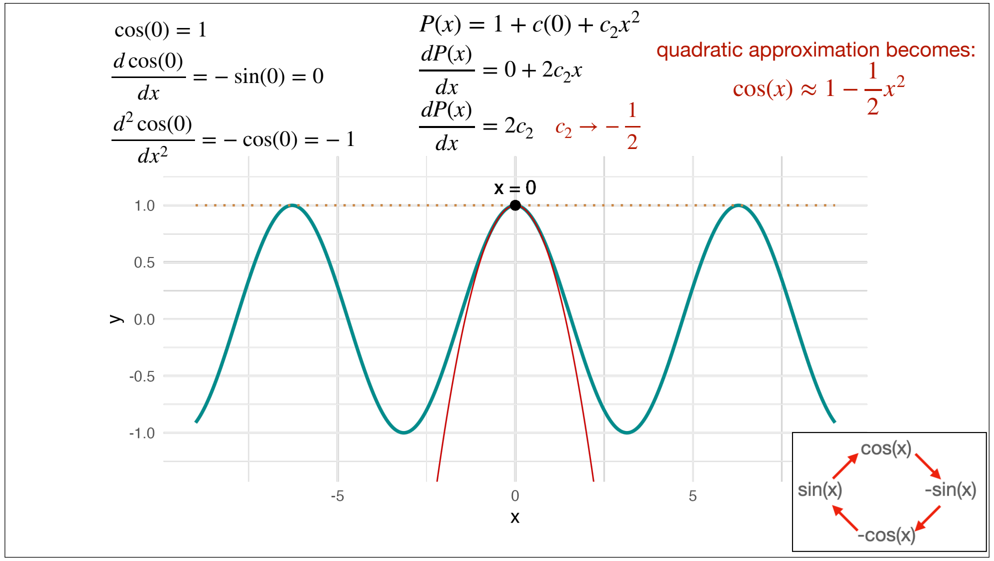

$\cos(0) = 1$
$\frac{d\cos(0)}{dx} = -\sin(0) = 0$
$\frac{d^2\cos(0)}{dx^2} = -\cos(0) = -1$
$P(x) = 1 + c(0) + c_2x^2$
$\frac{dP(x)}{dx} = 0 + 2c_2x$
$\frac{dP(x)}{dx} = 2c_2 \quad c_2 \to -\frac{1}{2}$
quadratic approximation becomes:
$\cos(x) \approx 1 - \frac{1}{2}x^2$
x = 0
y
x
cos(x)
sin(x)
-sin(x)
-cos(x)

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so far we've have 3 degrees of freedom
with our quadratic approximation

$P(x) = c_0 + c_1 x + c_2 x^2$

quadratic approximation becomes:

$\cos(x) \approx 1 - \frac{1}{2}x^2$

controls $P(0)$ — controls $\frac{dP}{dx}(0)$ — controls $\frac{d^2P}{dx^2}(0)$

$$P(x) = c_0 + c_1 x + c_2 x^2$$

- c_0: make sure the output of the approximation matches that of cosine x at $x = 0$
- c_1: was in charge of making sure that the derivatives match at that point
- c_2: make sure the second derivatives match up

more control of the approximation with
<u>more terms in the polynomial and matching higher order derivatives</u>

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$\vdots$

$\frac{d^2\cos(0)}{dx^2} = -\cos(0) = -1$

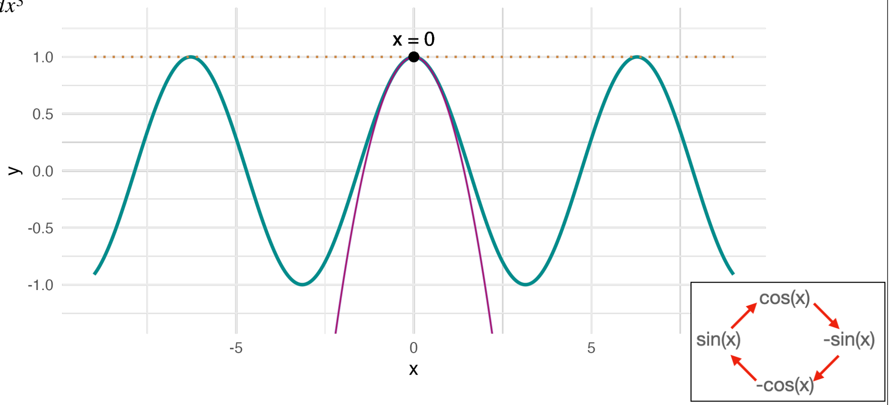
$\frac{d^3\cos(0)}{dx^3} = \sin(0) = 0$

$P(x) = 1 - \frac{1}{2}x^2 + c_3 x^3$

$\frac{d^3P(x)}{dx^3} = 0 + 1\cdot 2\cdot 3c_3x^0$
$= 1\cdot 2\cdot 3c_3 \quad c_3 \to 0$

cubic approximation becomes:

$\cos(x) \approx 1 - \frac{1}{2}x^2$



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$\vdots$

$\frac{d^3\cos(0)}{dx^3} = \sin(0) = 0$

$\frac{d^4\cos(0)}{dx^4} = \cos(0) = 1$

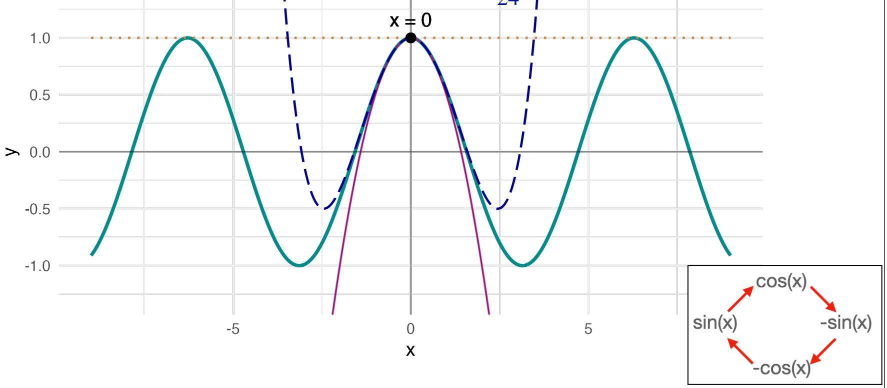
$P(x) = 1 - \frac{1}{2}x^2 + c_4 x^4$

$\frac{d^4P(x)}{dx^4} = 0 + 1\cdot 2\cdot 3\cdot 4c_3x^0$
$= 24\cdot c_4$

$c_4 \to \frac{1}{24}$

quartic approximation becomes:

$\cos(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$



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notice the power rule at play?

$\vdots$

$\frac{d^4}{dx^4}(c_4x^4) = 1\cdot 2\cdot 3\cdot 4\ c_4x^0$

$\frac{d^5}{dx^5}(c_5x^5) = 1\cdot 2\cdot 3\cdot 4\cdot 5c_5x^0$

$\frac{d^6}{dx^6}(c_6x^6) = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6c_6x^0$

$\frac{d^7}{dx^7}(c_7x^7) = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7c_7x^0$

$\vdots$

set $c_k = \frac{\text{desired derivative value}}{k!}$

quartic approximation becomes:

$\cos(x) \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$

$= 1 + (0)\frac{x^1}{1!} + (-1)\frac{x^2}{2!} + (0)\frac{x^3}{3!} + (1)\frac{x^4}{4!}$

Taylor polynomial of order 4 for $\cos(x)$

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example: exp(x)

$$\vdots$$
$$e^0 = 1$$
$$\downarrow d/dx$$
$$e^0 = 1$$
$$\downarrow d/dx$$
$$e^0 = 1$$
$$\downarrow d/dx$$
$$e^0 = 1$$
$$\downarrow d/dx$$
$$e^0 = 1$$
$$\vdots$$

$$P(x) = f(0) + \frac{df}{dx}\bigg|_{x=0}\frac{x^1}{1!} + \frac{d^2f}{dx^2}\bigg|_{x=0}\frac{x^2}{2!} + \frac{d^3f}{dx^3}\bigg|_{x=0}\frac{x^3}{3!} + \cdots$$

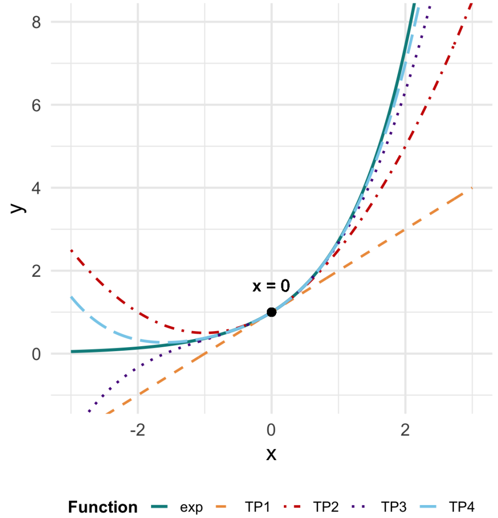
$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

$$= 1 + 1\frac{x^1}{1!} + 1\frac{x^2}{2!} + 1\frac{x^3}{3!} + 1\frac{x^4}{4!} + \cdots$$

Taylor polynomial of order 4 for e^x

example: exp(x)

$$P(x) = = 1 + 1\frac{x^1}{1!} + 1\frac{x^2}{2!} + 1\frac{x^3}{3!} + 1\frac{x^4}{4!} + \cdots$$



note: we can do this around every point and not just $x = 0$

Taylor's theorem

A function $f(x)$ can be expressed as a sum of terms derived from its derivatives at a specific point, plus a remainder term. Mathematically, the theorem can be written as:

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\text{Taylor polynomial of degree } n} + R_n(x)$$

where

a is the point around which the function is approximated

n is the order of the polynomial approximation

$R_n(x)$ is the remainder term, representing the error in the approximation after n terms.

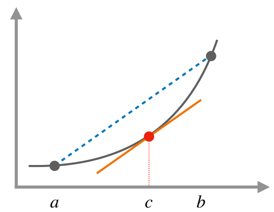
mean value theorem

Suppose $f(x)$ is a continuous function on closed interval $[a, b]$ and is differentiable on the interval's interior (a, b) at which a point c exists such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

slope of the tangent line → $f'(c)$

slope of the secant line between $x = a$ and $x = b$ → $\frac{f(b) - f(a)}{b - a}$



mean value theorem

Suppose $f(x)$ is a continuous function on closed interval $[a, b]$ and is differentiable on the interval's interior (a, b) at which a point c exists such that

slope of the tangent line → $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ ← slope of the secant line between $x = a$ and $x = b$

we can re-write the above equation as

$$f(b) = f(a) + f'(c)(b - a)$$
$$\approx f(a) + f'(a)(b - a)$$

this looks very much like the linear approximation for $f(b)$ using the tangent line at $x = a$

this is the idea behind Taylor's Theorem and using polynomials to approximate a smooth function: MVT lets us rewrite $f(b)$ in a form that resembles the first-term Taylor approximation

Taylor approximation

example

Find the Taylor polynomial of $f(x) = \dfrac{1}{1+x}$ of degree 3 at $x = 0$.

We have that $TP_3(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2}x^2 + \dfrac{f'''(0)}{3!}x^3$

Computing the successive derivatives we get

$$f(x) = \frac{1}{1+x}, \quad f'(x) = \frac{-1}{(1+x)^2}, \quad f''(x) = \frac{2}{(1+x)^3}, \quad f'''(x) = \frac{-6}{(1+x)^4}$$

Substituting $x = 0$ we get

$$f(0) = 1, \quad f'(0) = -1, \quad f''(0) = 2, \quad f'''(0) = -6$$

Therefore, the Taylor polynomial of f of degree 3 at $x = 0$ is equal to

$$TP_3(x) = 1 - x + x^2 - x^3$$

Taylor approximation

exercise 2

Approximate the function $f(x) = \sqrt{1+x}$ using a Taylor polynomial of degree 3 centered at $x = 0$. Compare it to the true value.