Introduction to Linear Algebra
Fundamental Vector Operations
Lecture 10

Termeh Shafie

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what is linear algebra?

Linear

- having to do with lines, planes, space, etc.
- example: $x + y + 3z = 7$ (and not $\sin, \log, x^2, \ldots$)

Algebra

- solving equations involving numbers and symbols
- *"reunion of broken parts"*

Linear algebra is the math of vectors and matrices, so we'll start by definitions and the mathematical operations we can perform on vectors and matrices.

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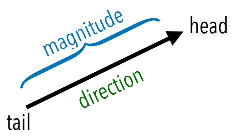
why do we need linear algebra?

- Coordinates and Geometry:
  Linear algebra provides a way to represent and understand geometric concepts like lines, planes, and transformations in higher dimensions.
- Equations Systems:
  It offers tools to solve systems of linear equations, which appear in countless areas of study.

3

what is a vector?

- object that has both a magnitude and a direction (physics)
- a list of numbers (data science)
- vector is rooted at origin (0,0)
- components: the entries of a vector
- the dimension of a vector is the number of components in the vector.
- notation: bold face ($\mathbf{x}, \mathbf{y}, \mathbf{z}$) or arrow over ($\vec{x}, \vec{y}, \vec{z}$)



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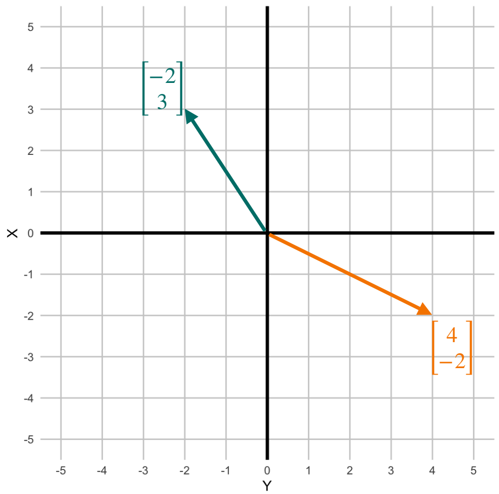
what is a vector?



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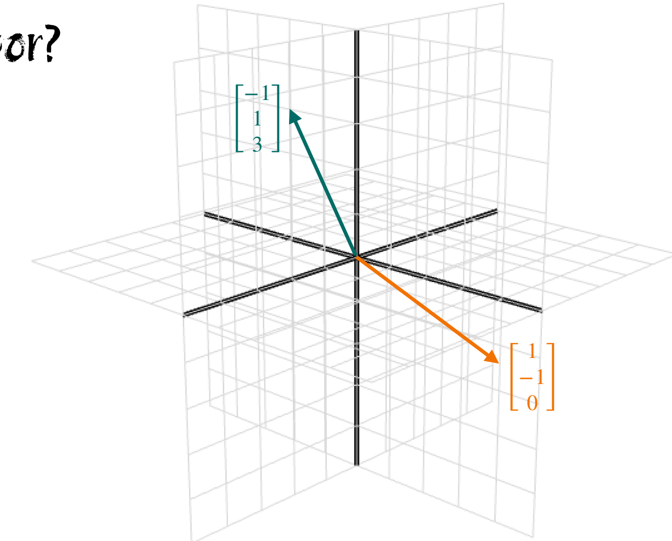

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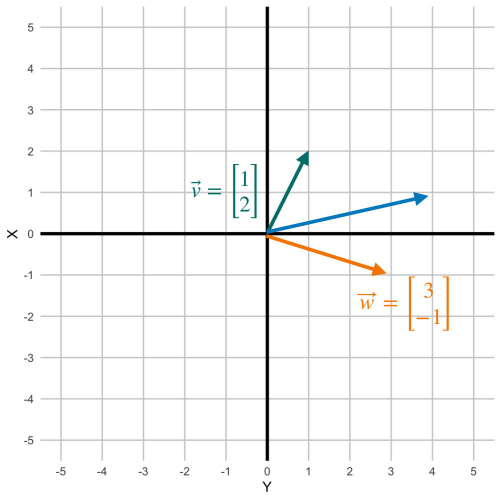
vector operations

vector addition

$$\vec{v} + \vec{w} = \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\-1\end{bmatrix}$$

$$= \begin{bmatrix}1+3\\2+(-1)\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix}$$

(same idea as when you add numbers on a number line)



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vector operations

scalar multiplication

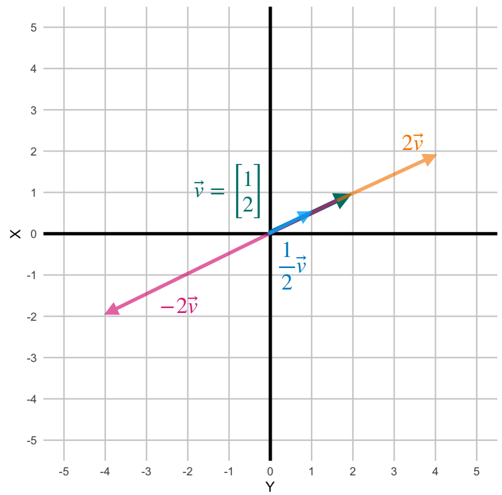
the process of stretching and squishing is called scaling

the scalars here are $2, \ -2$ and $\dfrac{1}{2}$

$$2\vec{v} = 2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$$

$$-2\vec{v} = -2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-2\\-4\end{bmatrix}$$

$$\frac{1}{2}\vec{v} = \frac{1}{2}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0.5\\1\end{bmatrix}$$



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magnitudes and direction

example

What is the magnitude of $\vec{a} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$?

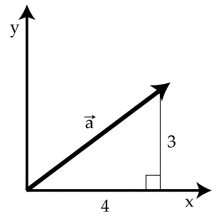
The magnitude of a vector is the distance from the endpoint of the vector to the origin, i.e. its length

This vector extends 4 units along the x-axis, and 3 units along the y-axis.

Magnitude $||\vec{a}||$ is computed using Pythagorean Theorem ($x^2 + y^2 = z^2$):

$$||\vec{a}|| = \sqrt{x^2 + y^2} = \sqrt{4^2 + 3^2} = 5$$

The magnitude of a vector is a scalar value.



basis vectors

the x coordinate of a vector can be viewed as a scalar scaling $\hat{\imath}$

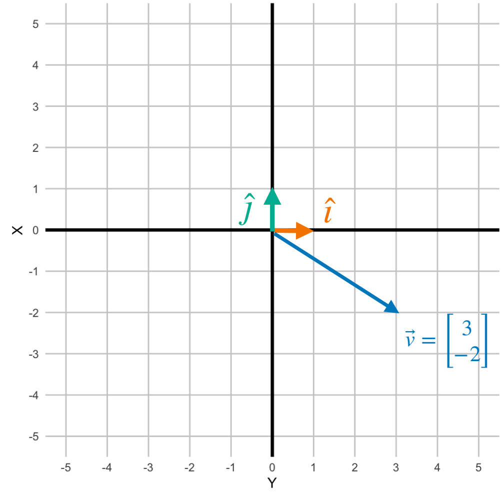
the y coordinate of a vector can be viewed as a scalar scaling $\hat{\jmath}$

$\hat{\imath}$ and $\hat{\jmath}$ are unit vectors: they have length 1

$$\vec{v} = (3)\hat{\imath} + (-2)\hat{\jmath}$$

$(\hat{\imath}, \hat{\jmath})$ are called the basis vectors of the xy coordinate system

Any vector in the plane can be written as a combination of these two vectors

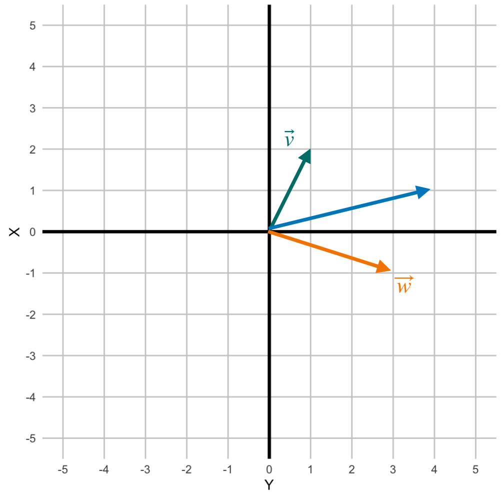


basis vectors

what if we chose some other basis vector?

the choice of basis vector effect the numerical values of the vectors

scaling vectors and adding them is a linear combination of those two vectors



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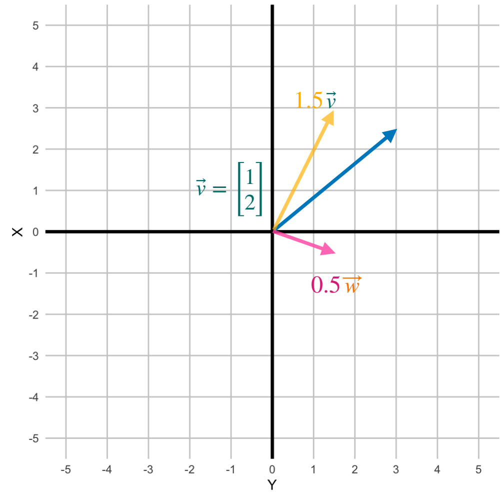

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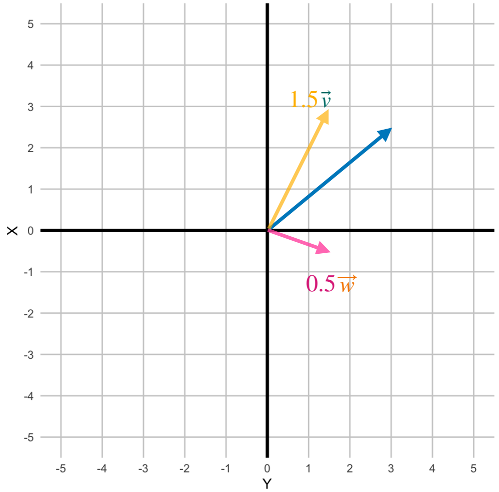
scaling vectors and adding them is called a linear combination of those two vectors

the span of the vector $\vec{v}$ and $\vec{w}$ is the set of all their linear combinations

$$a\vec{v} + b\vec{w}$$

where $a, b \in \mathbb{R}$.



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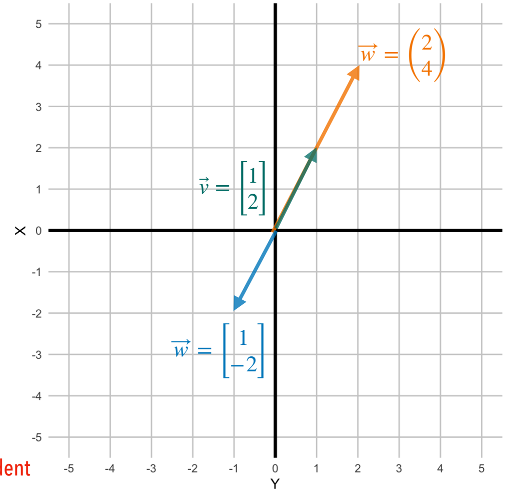
where $a, b \in \mathbb{R}$.

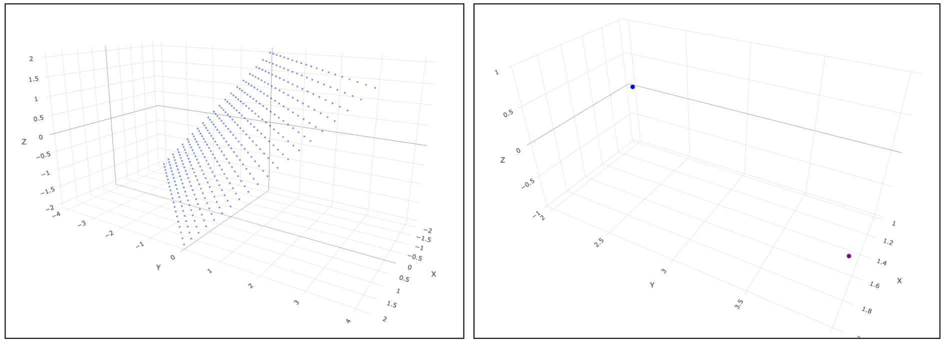
If $\vec{v}$ and $\vec{w}$ do not line up (they are not scalar multiples of each other), their span is a plane. If they do line up, their span is a line.

In the latter case the vectors are linearly dependent



linear combinations: planes and lines

- Scaling vectors and adding them is a linear combination of those two vectors
- Geometrically, the linear combinations of a nonzero vector form a line. The linear combinations of two nonzero vectors form a plane, unless the two vectors are collinear, in which case they form a line.



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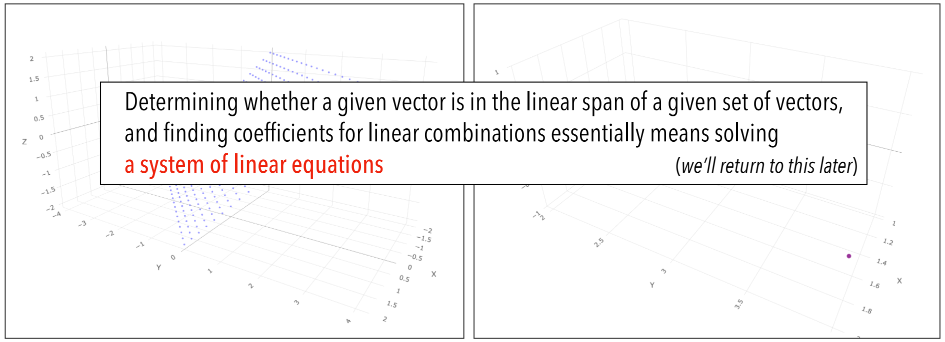
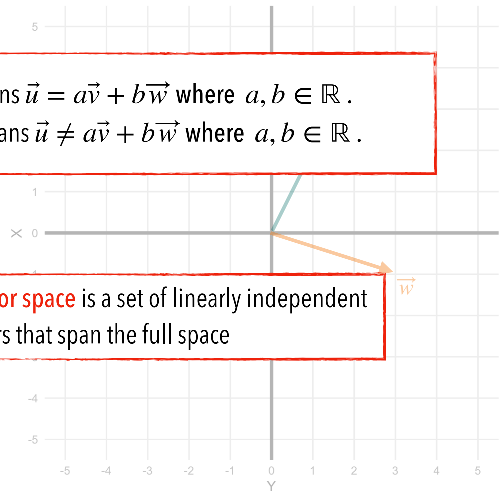


Determining whether a given vector is in the linear span of a given set of vectors, and finding coefficients for linear combinations essentially means solving a system of linear equations (*we'll return to this later*)

linear independence



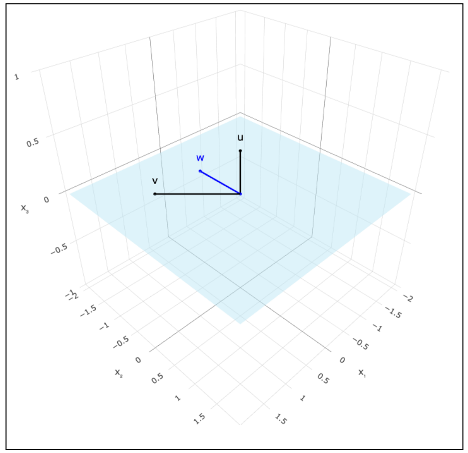
Linearly dependent means $\vec{u} = a\vec{v} + b\vec{w}$ where $a, b \in \mathbb{R}$.
Linearly independent means $\vec{u} \neq a\vec{v} + b\vec{w}$ where $a, b \in \mathbb{R}$.

the basis of the vector space is a set of linearly independent vectors that span the full space

linear independence and spanning vectors

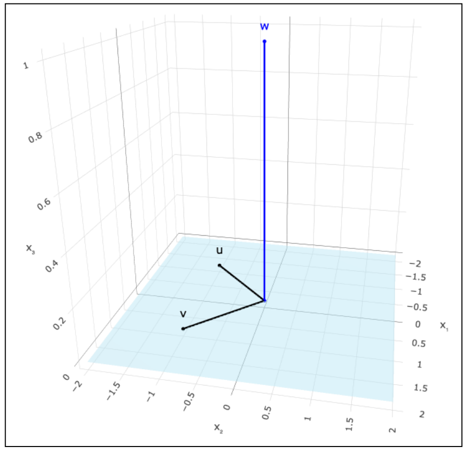
$\vec{w}$ is in $span(\vec{u}, \vec{v})$or the plane spanned by $(\vec{u}, \vec{v})$

$\vec{w}$ is a linear combination of $(\vec{u}, \vec{v})$, so $(\vec{u}, \vec{v}, \vec{w})$ is not linear independent.

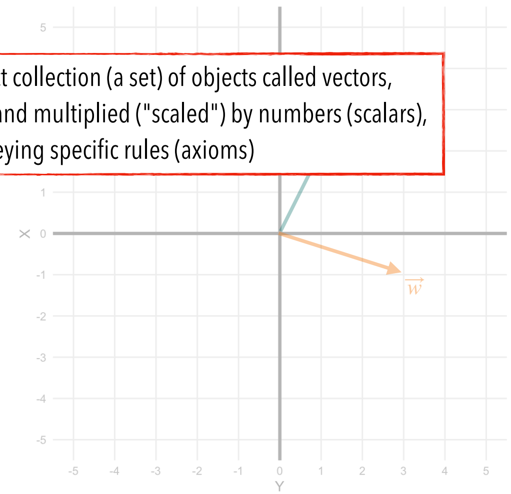


linear independence and spanning vectors

Since $\vec{w}$ is not in $span(\vec{u}, \vec{v})$, $(\vec{u}, \vec{v}, \vec{w})$ is linear independent.



the vector space



A vector space is an abstract collection (a set) of objects called vectors, which can be added together and multiplied ("scaled") by numbers (scalars), all while obeying specific rules (axioms)

the vector space

The 'rules' (axioms) to hold true for all vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and scalars a, b are the following

- Closure: $\vec{u} + \vec{v} \in V, a\vec{v} \in V$
- Commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- Associativity: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- Identity for Addition: There is a zero vector $\vec{0} \in V$ such that $\vec{u} + \vec{0} = \vec{u}$
- Distributive Property: $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ and $(a + b)\vec{u} = a\vec{u} + b\vec{u}$
- Associativity of Scalars: $(ab)\vec{u} = a(b\vec{u})$
- Multiplicative Identity: $1\vec{u} = \vec{u}$

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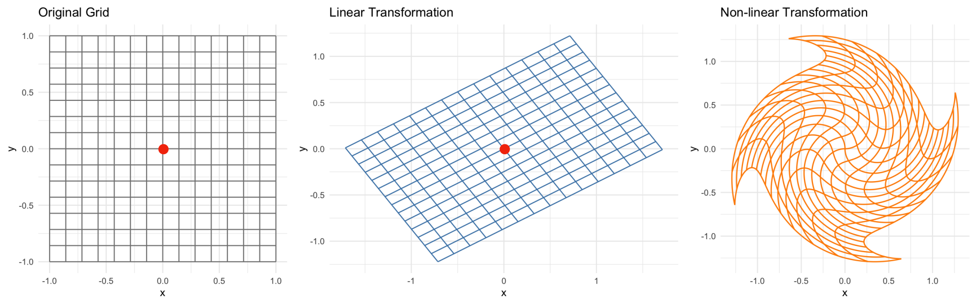
linear transformations & matrices



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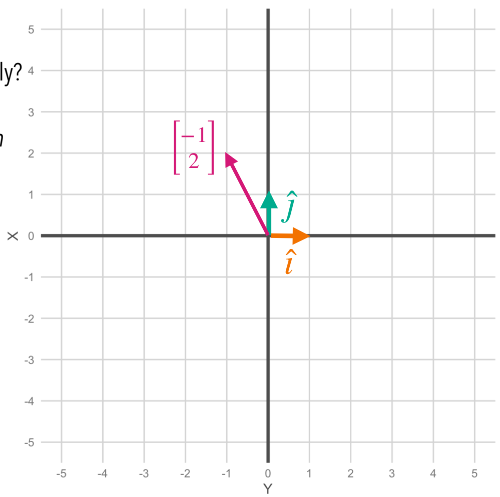
linear transformations

how to describe linear transformations numerically?

by using unit vectors $\hat{\imath}$ and $\hat{\jmath}$ and keeping track of where they land after the transformation

example

$\vec{v} = -1\hat{\imath} + 2\hat{\jmath}$



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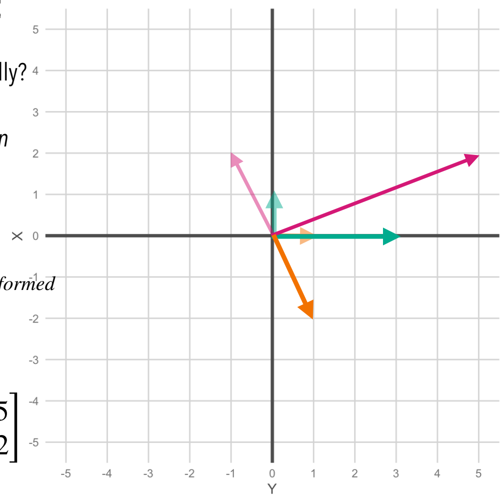
$\vec{v} = -1\hat{\imath} + 2\hat{\jmath}$

$$\vec{v}_{transformed} = -1\hat{\imath}_{transformed} + 2\hat{\jmath}_{transformed}$$

$$= -1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1(1) + 2(3) \\ -1(-2) + 2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$



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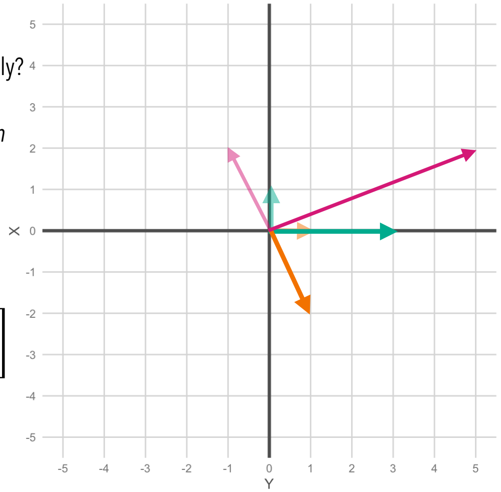
generally

$\hat{\imath} \rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \hat{\jmath} \rightarrow \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow x \begin{bmatrix} 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$$



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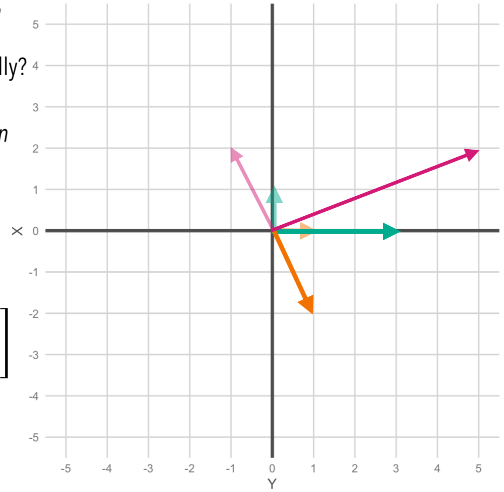
any 2D linear transformation is completely described using only these four numbers

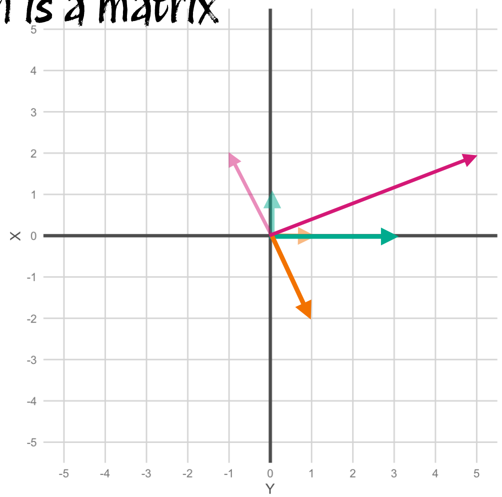
let's put them in a 2x2 grid aka a matrix



a linear transformation is a matrix

$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$$

where $\hat{\imath}$ lands **where $\hat{\jmath}$ lands**



matrix as a transformation of space

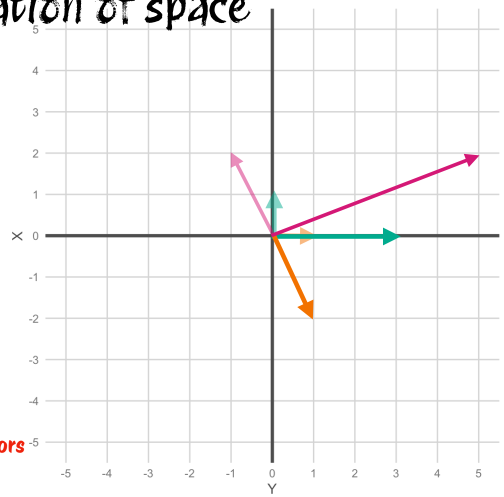
$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

any vector

$$\Longrightarrow x \begin{bmatrix} 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

adding scaled versions of our new basis vectors



matrix as a transformation of space

generally:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\implies x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{matrix vector multiplication}}$$

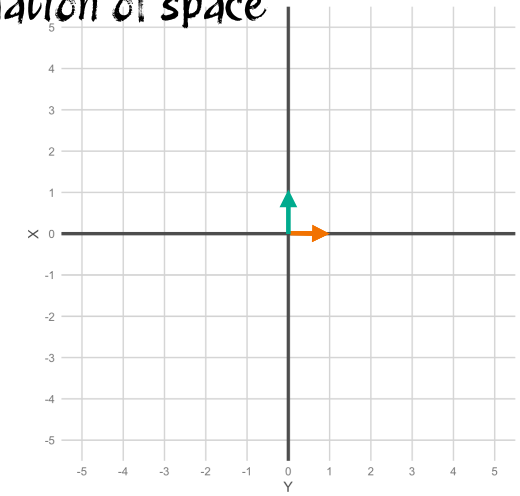
matrix as a transformation of space

example

rotate space 90° counterclockwise

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

where $\hat{i}$ lands   where $\hat{j}$ lands



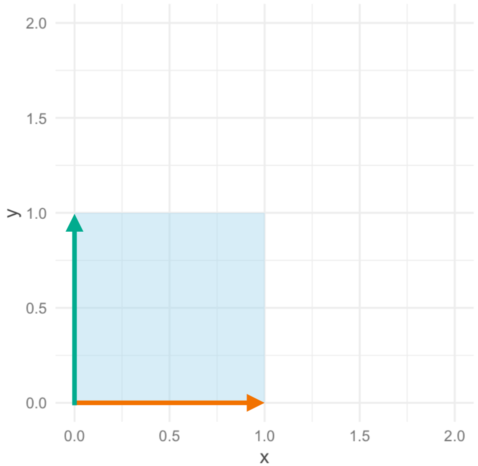
matrix as a transformation of space

example

the identity matrix (no transformation)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1x+0y \\ 0x+1y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

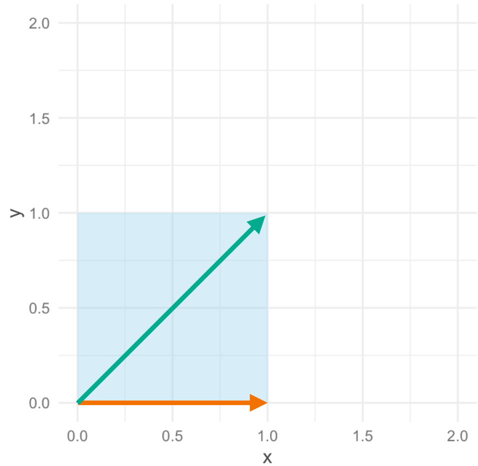


matrix as a transformation of space

example

a shear transformation

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



matrix as a transformation of space

example

a shear transformation

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$= \begin{bmatrix}1x + 1y\\0x + 1y\end{bmatrix} = \begin{bmatrix}x + y\\y\end{bmatrix}$$