Matrix Arithmetics & Matrix Properties

Lecture 11

Termeh Shafie

1

recap

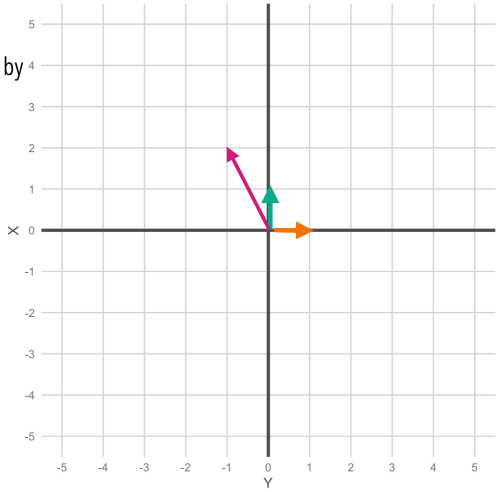
a linear transformation is completely determined by where it takes the basis vectors of the space

in 2D, this is $\hat{\imath}$ and $\hat{\jmath}$

why?

because any other vector can be described as as linear combination of thesis basis vectors:

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \vec{v} = x\hat{\imath} + y\hat{\jmath}$$



2

recap

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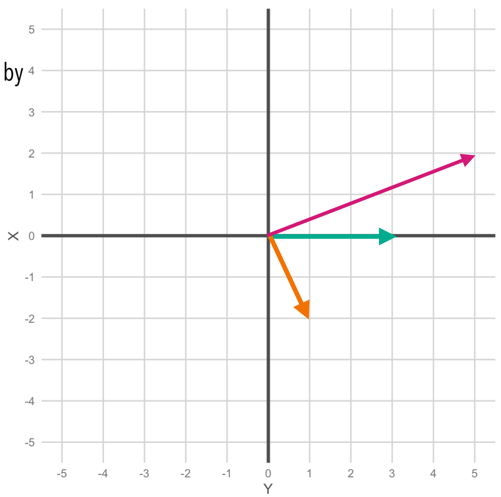
$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \vec{v} = x\hat{\imath} + y\hat{\jmath}$$

after transformation T :

$$T(\vec{v}) = xT(\hat{\imath}) + yT(\hat{\jmath})$$



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recap

generally:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Longrightarrow x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{matrix vector multiplication}}$$

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the matrix

- A matrix is a table of numbers rather than a list as is the case for vectors
- The size of a matrix: number of rows × number of columns = $m \times n$ (read "m *by* n")

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A square matrix is a matrix that has an equal number of columns and rows, i.e., $m = n$
- A zero matrix is a square matrix in which all elements are 0

the matrix

- A diagonal matrix is a square matrix with non-zero elements only on the main diagonal
- An identity matrix is a diagonal matrix in which all elements on the main diagonal are 1:

$$D_{n\times n} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad I_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The identity matrix is special because, when multiplied by another matrix, it produces the original matrix back again (we'll return to this later after covering matrix multiplication)
- A lower triangular matrix has non-zero elements only on or below the main diagonal
- An upper triangular matrix has non-zero elements only on or above the main diagonal
- A symmetric matrix is a square matrix with elements symmetric such that $a_{ij} = a_{ji}$

the transpose of a matrix

Let A be an $m \times n$ matrix. The transpose of A, denoted A^T or A', is the $n \times m$ matrix whose columns are the respective rows of A.

- A matrix is symmetric if it doesn't change when you take its transpose

example

If you take the transpose of matrices $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

we get $A^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$ and $B^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.

Note: matrix B is thus symmetric.

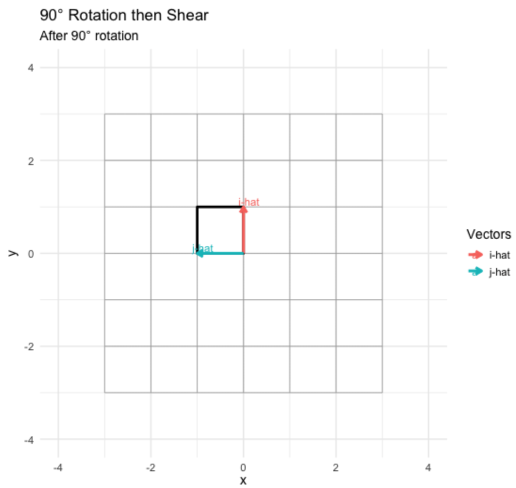
composition

example: first rotate 90° and then apply a shear

what is the final landing spot for $\hat{\imath}$ and $\hat{\jmath}$?

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

composition



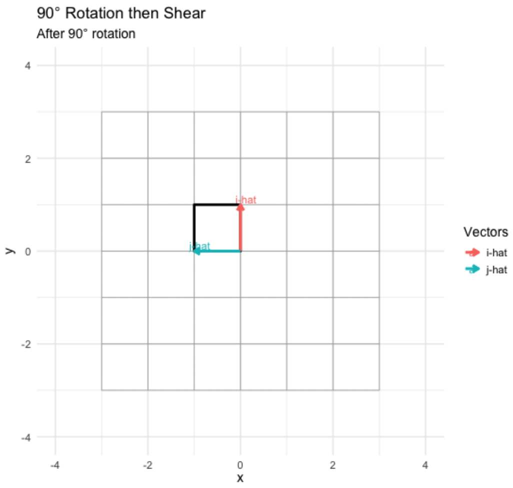
composition

example: first rotate 90° and then apply a shear

what is the final landing spot for $\hat{\imath}$ and $\hat{\jmath}$?

$$\underbrace{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}}_{\text{shear}}\left(\underbrace{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}}_{\text{rotate}}\begin{bmatrix}x\\y\end{bmatrix}\right)$$

$$= \underbrace{\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}}_{\text{composition}}\begin{bmatrix}x\\y\end{bmatrix}$$



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composition

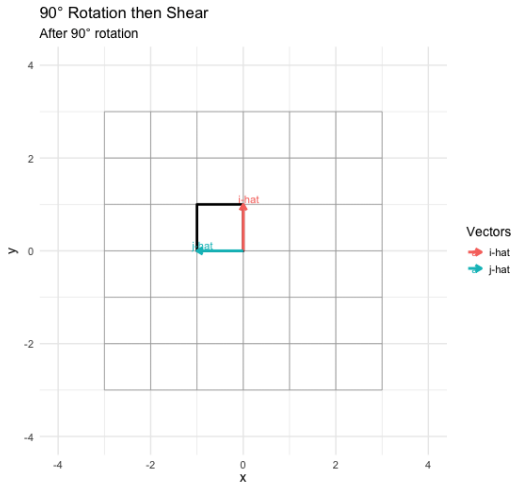
example: first rotate 90° and then apply a shear

what is the final landing spot for $\hat{\imath}$ and $\hat{\jmath}$?

$$\underbrace{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}}_{\text{shear}}\underbrace{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}}_{\text{rotate}} = \underbrace{\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}}_{\text{composition}}$$

read right to left just as function notation

the product of the two transformation matrices



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composition

example: first rotate 90° and then apply a shear, let's compute the product

$$\underbrace{\overset{T_2}{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}}}_{\text{shear}}\underbrace{\overset{T_1}{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}}}_{\text{rotate}}$$

where does $\hat{\imath}$ go after T_1?
where does $\hat{\imath}$ go after T_2?

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

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composition

example: first rotate 90° and then apply a shear, let's compute the product

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where does $\hat{\jmath}$ go after T_1?
where does $\hat{\jmath}$ go after T_2?

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}-1\\0\end{bmatrix} = -1\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

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composition

example: first rotate 90° and then apply a shear, let's compute the product

$$\underset{\text{shear}}{\overset{T_2}{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}}}\underset{\text{rotate}}{\overset{T_1}{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}}}$$

where does $\hat{\imath}$ go after T_1?
where does $\hat{\imath}$ go after T_2?
$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

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$$\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}$$

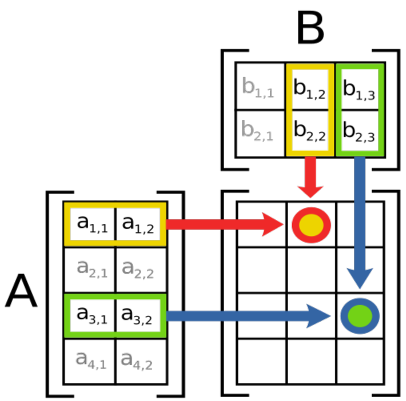
matrix multiplication

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}e & f\\g & h\end{bmatrix} = \begin{bmatrix}ae+bg & af+bh\\ce+dg & cf+dh\end{bmatrix}$$

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}e\\g\end{bmatrix} = e\begin{bmatrix}a\\c\end{bmatrix} + g\begin{bmatrix}b\\d\end{bmatrix} = \begin{bmatrix}ae+bg\\ce+dg\end{bmatrix}$$ **where $\hat{\imath}$ lands**

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}f\\h\end{bmatrix} = f\begin{bmatrix}a\\c\end{bmatrix} + h\begin{bmatrix}b\\d\end{bmatrix} = \begin{bmatrix}af+bh\\cf+dh\end{bmatrix}$$ **where $\hat{\jmath}$ lands**

matrix multiplication



https://commons.wikimedia.org/wiki/File:Matrix_multiplication_diagram_2.svg

matrix multiplication

Let A be an $m \times r$ matrix, and let B be an $r \times n$ matrix.

The matrix product of A and B, denoted $A \cdot B$ or AB, is the $m \times n$ matrix M whose entry in the ith row and jth column is the product of the ith row of A and the jth column of B.

- In order to multiply two matrices A and B, the number of columns of A must be the same as the number of rows of B (the inner dimensions must be the same)
- The resulting matrix has same number of rows as A and same number of columns as B (i.e. the outer dimensions)

can you see why?

$$\overbrace{(m \times \underbrace{r) \times (r}_{\text{inner dimensions must match}} \times n)}^{\text{final dimensions are outer dimensions}}$$

Non-Square Matrices?

$$\begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix}$$

where $\hat{\imath}$ lands   where $\hat{\jmath}$ lands

transformations between dimensions

$$\begin{bmatrix} 2 \\ 7 \end{bmatrix} \longrightarrow T(\vec{v}) \longrightarrow \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$$

2D input   3D output

Non-Square Matrices?

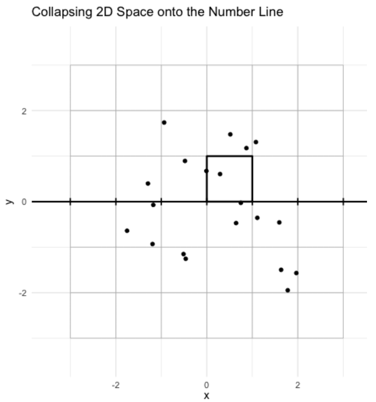
transformations between dimensions

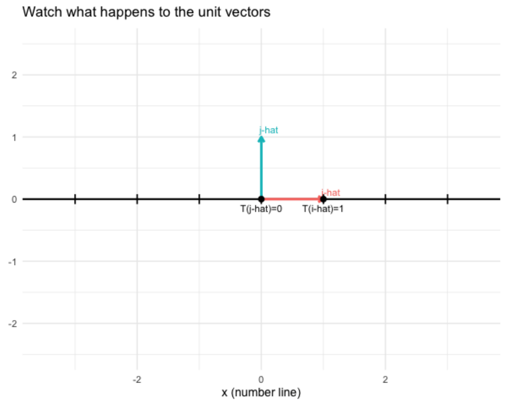
$$\begin{bmatrix} 2 \\ 7 \end{bmatrix} \longrightarrow T(\vec{v}) \longrightarrow [1.8]$$

2D input   1D output

Non-Square Matrices?

a projection that collapses everything onto one axis (the number line): $\mathbb{R}^2 \to \mathbb{R}$





vector multiplication

Two principal ways of multiplying vectors:

1. Dot products (a.k.a. scalar products)

   $d = \vec{a} \cdot \vec{b}$

   generates a scalar value from the product of two vectors

2. Cross products

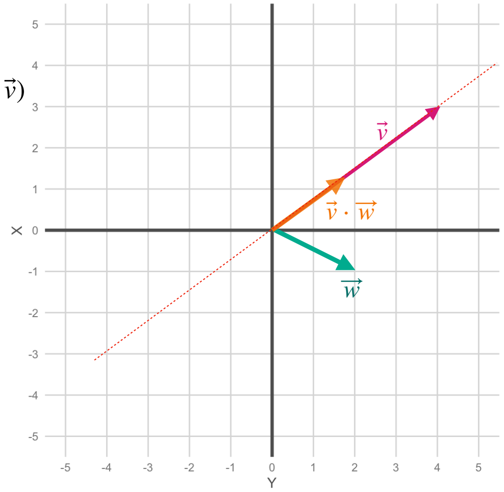
   $\vec{c} = \vec{a} \times \vec{b}$

   The cross product generates a vector from the product of two vectors

dot product

$\vec{v} \cdot \vec{w} =$ (length of projected $\vec{w}$) $\cdot$ (length of $\vec{v}$)

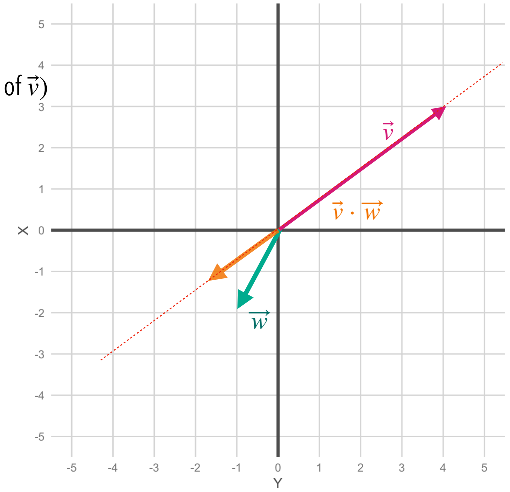
$\vec{v} \cdot \vec{w} > 0$ (same direction)



dot product

$\vec{v} \cdot \vec{w} = -$ (length of projected $\vec{w}$) $\cdot$ (length of $\vec{v}$)

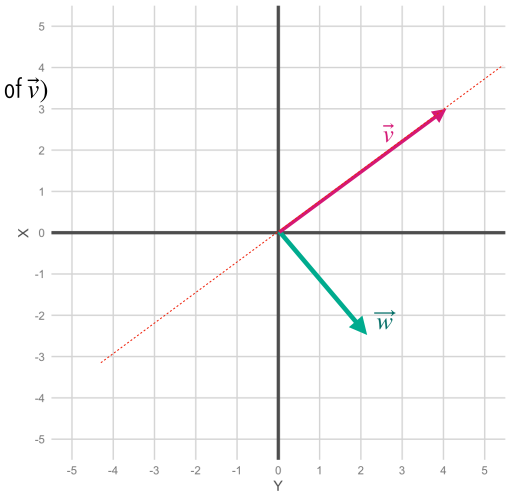
$\vec{v} \cdot \vec{w} < 0$ (opposite direction)



dot product

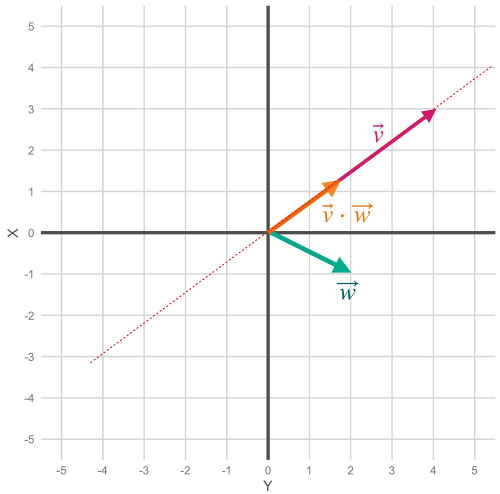
$\vec{v} \cdot \vec{w} = -$ (length of projected $\vec{w}$) $\cdot$ (length of $\vec{v}$)

$\vec{v} \cdot \vec{w} = 0$ (perpendicular)



dot product

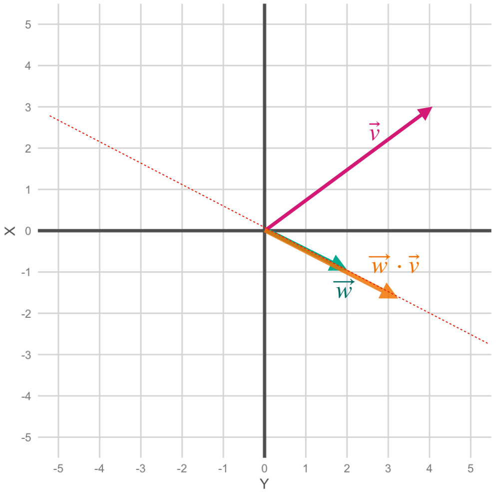
but does order mater?



dot product

but does order mater?

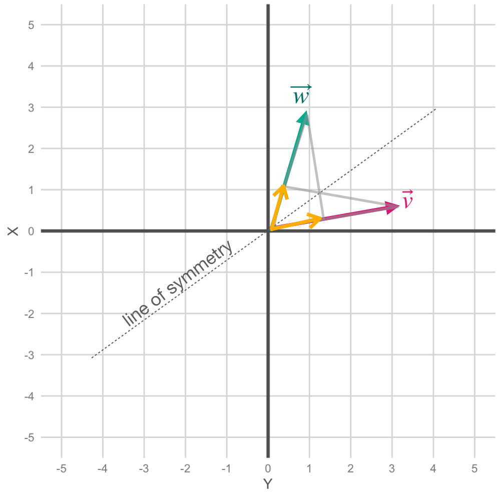
NO!



dot product

but does order mater?

think of it this way…..



dot product

The dot product is key for calculating vector projections, vector decompositions, and determining orthogonality

The dot product of two vectors $\vec{a}$ and $\vec{b}$ is

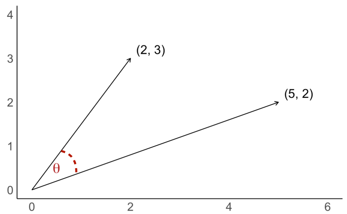
$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

The angle θ of between two vectors is determined by the formula

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \, ||\vec{b}|| \cos \theta$$

where $||\vec{a}||$ is the length or norm or magnitude of a vector. Thus

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}|| \, ||\vec{b}||} = \frac{\vec{a}}{||\vec{a}||} \cdot \frac{\vec{b}}{||\vec{b}||}$$

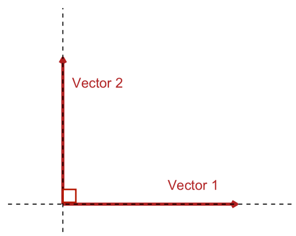


orthogonality

- $\cos 0^\circ = 1$ the vectors point in exactly the same direction (they coincide)
- $\cos 90^\circ = 0$ means the vectors are perpendicular (aka orthogonal) to each other in 2D or 3D

Two vectors are orthogonal to one another if the dot product of those two vectors is equal to zero

- Orthogonal vectors point in completely independent directions, meaning one vector cannot be expressed as a scalar multiple of the other



orthogonality

Two vectors are orthogonal to one another if the dot product of those two vectors is equal to zero

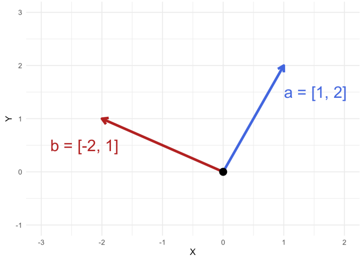
example

Let $\vec{a} = (1,2)$ and $\vec{b} = (-2,1)$

The dot product is

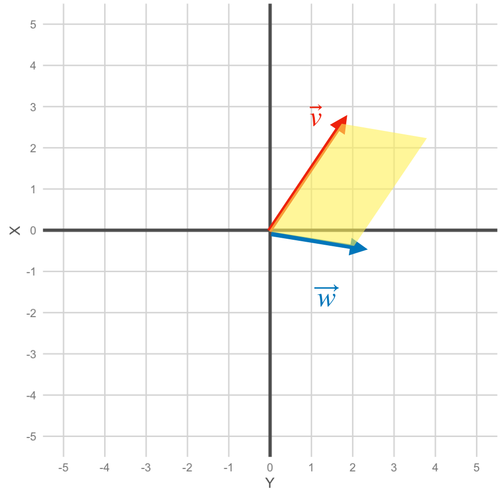
$$\vec{a} \cdot \vec{b} = (1)(-2) + (2)(1) = -2 + 2 = 0$$

Since their dot product is zero, $\vec{a}$ and $\vec{b}$ are orthogonal.



cross product

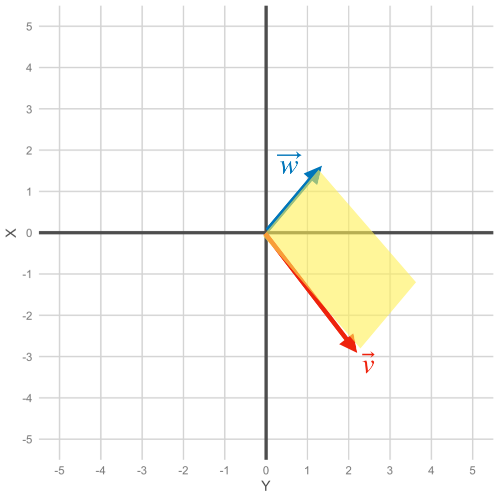
$\vec{v} \times \vec{w}$ = signed area of parallelogram



cross product

$\vec{v} \times \vec{w}$ = signed area of parallelogram

if $\vec{v}$ is to the right of $\vec{w}$ the area is positive

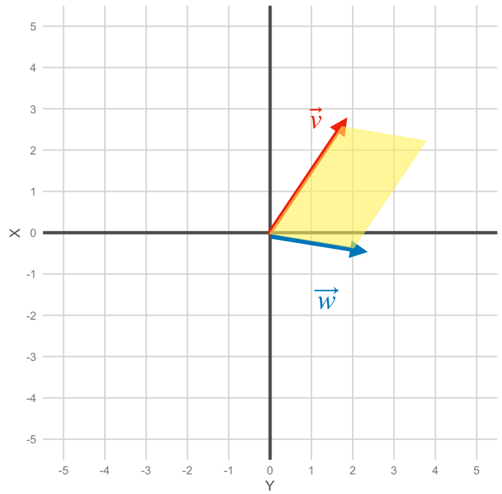


cross product

$\vec{v} \times \vec{w}$ = signed area of parallelogram

if $\vec{v}$ is to the right of $\vec{w}$ the area is positive

if $\vec{v}$ is to the left of $\vec{w}$ the area is negative



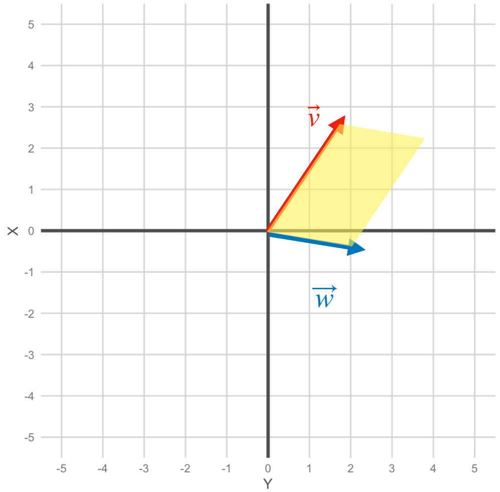
cross product

$\vec{v} \times \vec{w} =$ **signed** area of parallelogram

if $\vec{v}$ is to the right of $\vec{w}$ the area is positive
if $\vec{v}$ is to the left of $\vec{w}$ the area is negative

order matters!

$$\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$$



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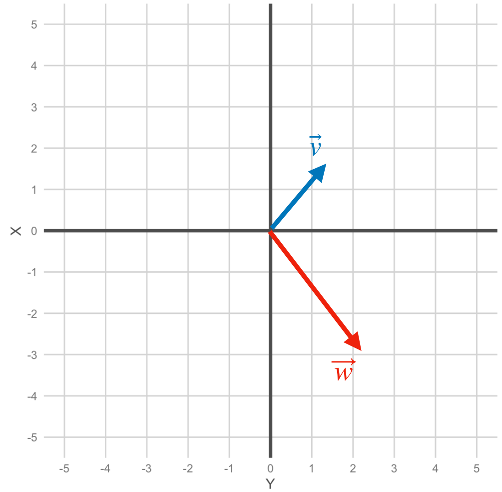
cross product

$\vec{v} \times \vec{w} =$ **signed** area of parallelogram

if $\vec{v}$ is to the right of $\vec{w}$ the area is positive
if $\vec{v}$ is to the left of $\vec{w}$ the area is negative

example

$\vec{v} \times \vec{w} = -A$ because $\vec{v}$ is to the left of $\vec{w}$



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cross product and determinant

to compute the cross product we need to introduce
the determinant of a matrix

recall that linear transformations
stretch or squishes space, but there question is
how much is the space stretched or squished?

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determinant

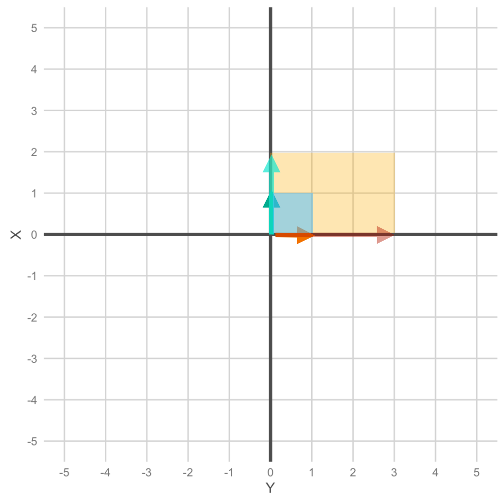
the determinant of a matrix

recall that linear transformations
stretch or squishes space, but there question is
how much is the space stretched or squished?

example

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

new area $= 3 \times 2 = 6$



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determinant

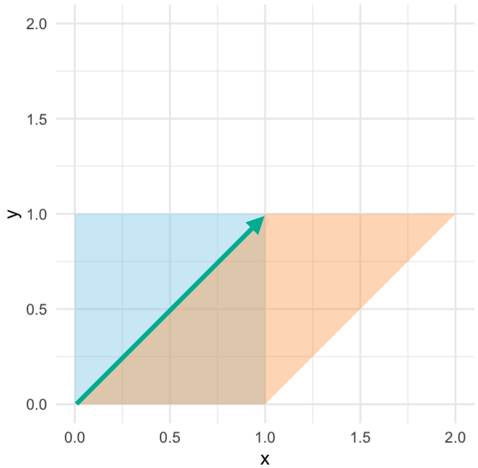
the determinant of a matrix

recall that linear transformations
stretch or squishes space, but there question is
how much is the space stretched or squished?

example

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ shear

new area $= 1 \times 1 = 1$



determinant

the determinant of a matrix

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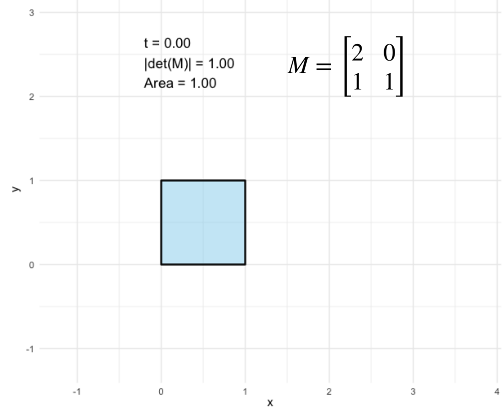

determinant

the determinant of a matrix

recall that linear transformations
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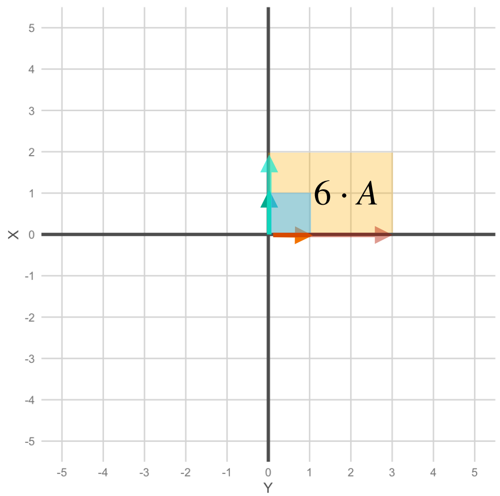
how much is the space stretched or squished?

example

$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

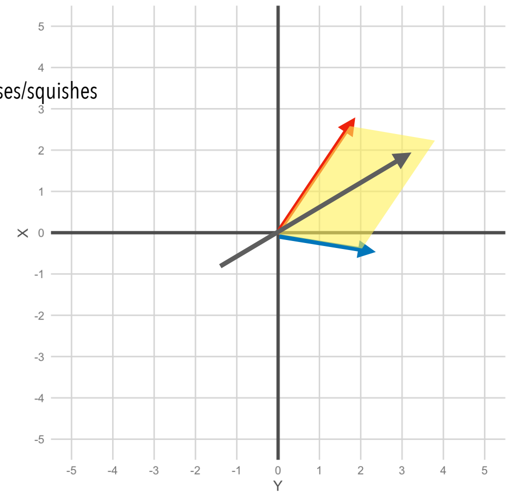
new area $= 3 \times 2 = 6$

$$= \det\left(\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\right)$$



determinant

determinant is > 1 when area increases/stretches
determinant is between 0 and1 when area decreases/squishes
when is determinant equal to 0?



determinant

determinant is > 1 when area increases/stretches

determinant is between 0 and1 when area decreases/squishes

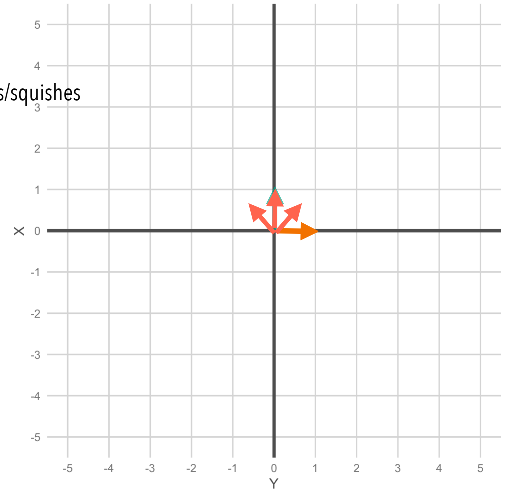
when is determinant equal to 0?

when the space is transformed into a line or point

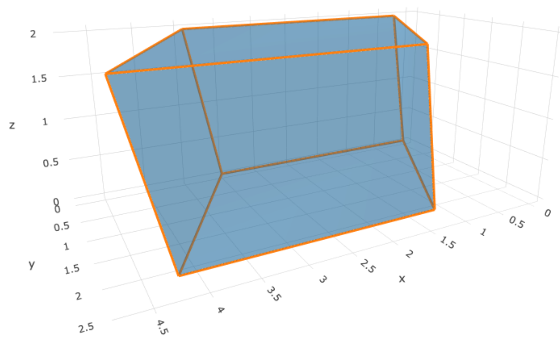
this is a very important result!

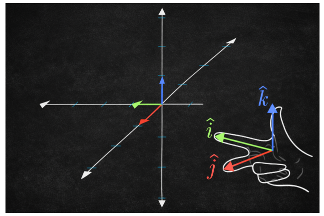
note: we can also have negative determinants which is then about orientation or "flipping" space

the absolute value will tell us the same thing though



determinant in 3D: parallelepiped





computing the determinant

Let A be an 2×2 matrix given as $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The determinant of A, denoted by

$$det(A) \text{ or } |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is given by $ad - bc$.

All good, but what if $n > 2$?

Then we need to define matrix minor and matrix cofactor.

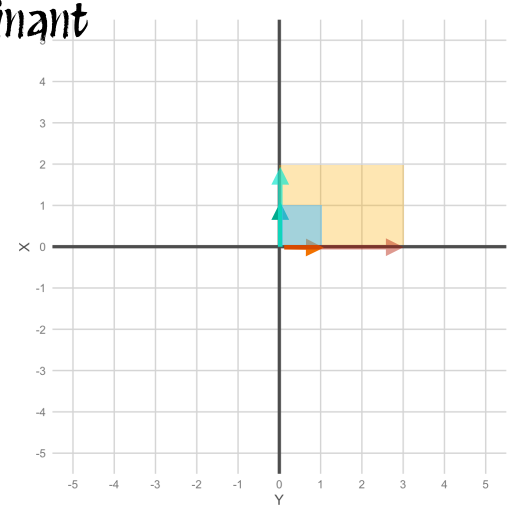
computing the determinant

example

$$\det\left(\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\right) = 6$$

intuition:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$



computing the determinant

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All good, but what if $n > 2$?

Then we need to define matrix minor and matrix cofactor.

computing the determinant: Laplace expansion

Let A be an 3×3 matrix given as $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

The determinant of A is given by

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

matrix minor and matrix cofactor

Let A be an $n \times n$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The i, j minor of A denoted $M_{i,j}$ is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the ith row and jthcolumn of A.

The i, j-cofactor of A is the number. $C_{ij} = (-1)^{i+j} M_{i,j}$

Let A be an $n \times n$ matrix where $n > 2$. Then $det(A)$ is the number found by taking the cofactor expansion along the first row of A. That is,

$$\det(A) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + \cdots + a_{1,n}C_{1,n}.$$

matrix minor and matrix cofactor

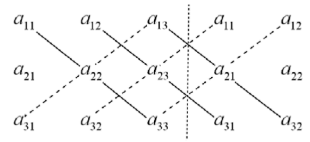
exercise

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(a) Find the cofactor expansions along the first column.

(b) Find the determinant of A.

Note: during your tutorial you will cover another way to find the determinant called the butterfly method (only works for 3×3 matrices)



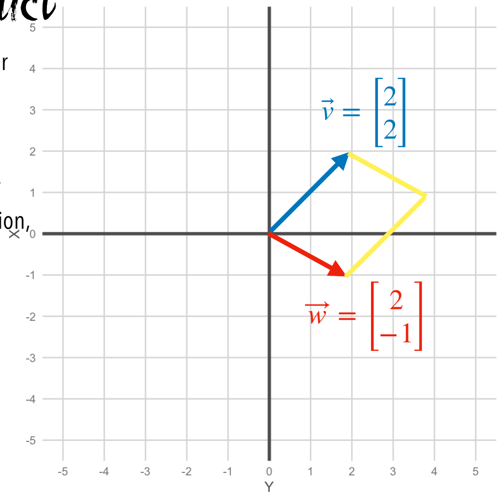
back to the cross product

Geometrically, the cross product represents a vector perpendicular to two given vectors whose length equals the area of the parallelogram they span.

The determinant provides the algebraic machinery behind the cross product: it encodes area, orientation, and perpendicular direction in a compact formula.

$$\vec{v} \times \vec{w} = \det\left(\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\right)$$

$$= 2 \cdot -1 - 2 \cdot 2 = -2$$



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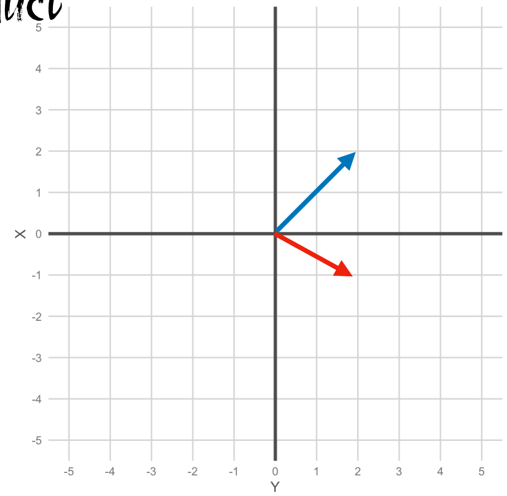
back to the cross product

- when vectors are (almost) perpendicular their area is larger



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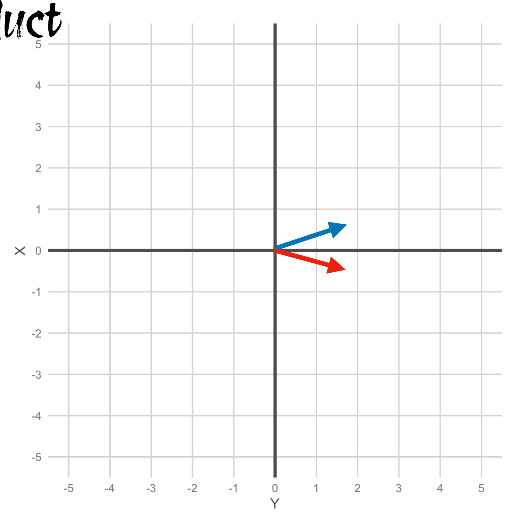
back to the cross product

- when vectors are (almost) perpendicular their area is larger
- when vectors are pointing in similar directions their area is smaller



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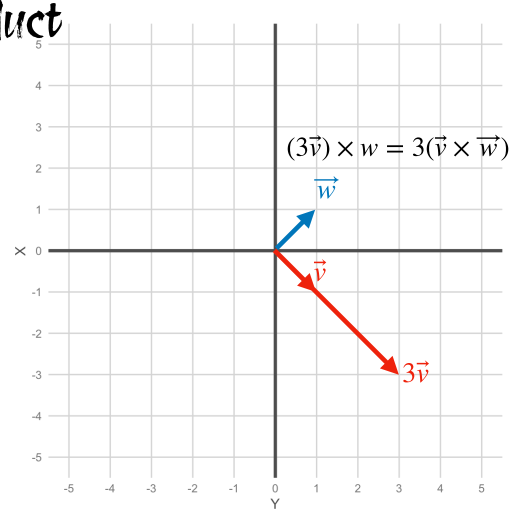
back to the cross product

- when vectors are (almost) perpendicular their area is larger
- when vectors are pointing in similar directions their area is smaller
- scaling up one of those vectors, increases the area by the same scalar



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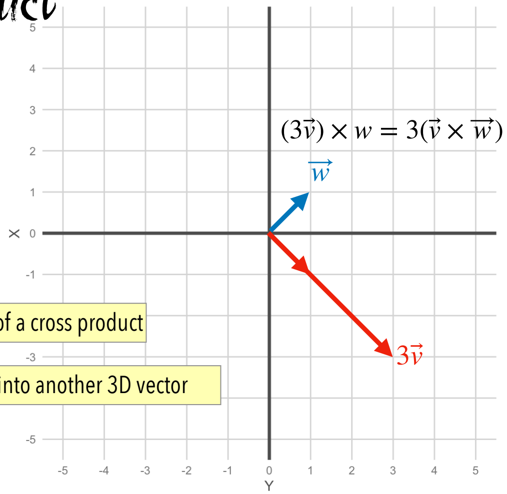
back to the cross product

- when vectors are (almost) perpendicular their area is larger
- when vectors are pointing in similar directions their area is smaller
- scaling up one of those vectors, increases the area by the same scalar

note: nothing so far is technically the definition of a cross product

cross product combines two different 3D vectors into another 3D vector



cross product: formally

Assume $\vec{v} \times \vec{w} = 3$

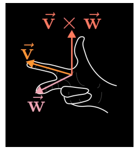
i.e. the area of the parallelogram

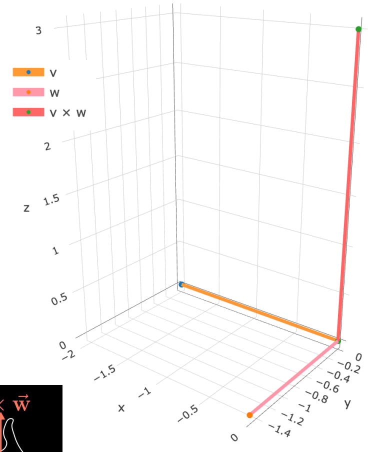
$\vec{v} \times \vec{w} = \vec{p}$

the cross product produces a vector and not a number

the new vector's **length** will be the area of the parallelogram

and the direction is going to be perpendicular to that parallelogram, but which way?





cross product: Laplace expansion

$$\vec{v} = \begin{bmatrix}-2\\0\\0\end{bmatrix}, \vec{w} = \begin{bmatrix}0\\-1.5\\0\end{bmatrix} \implies \text{Area} = \left|\det\begin{pmatrix}v_x & w_x\\ v_y & w_y\end{pmatrix}\right| = \left|\det\begin{pmatrix}-2 & 0\\ 0 & -1.5\end{pmatrix}\right|$$

$$\det = (-2)(-1.5) - (0)(0) = 3$$

$$\vec{v} \times \vec{w} = \begin{vmatrix}\hat{\imath} & -2 & 0\\ \hat{\jmath} & 0 & -1.5\\ \hat{k} & 0 & 0\end{vmatrix}$$

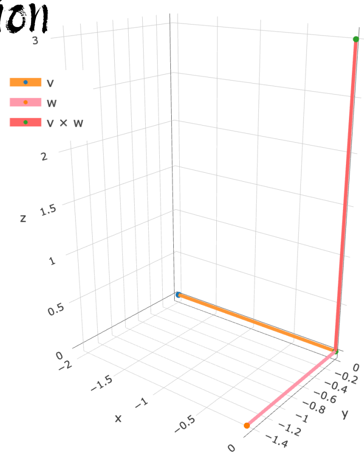
$$= \hat{\imath}\begin{vmatrix}0 & -1.5\\ 0 & 0\end{vmatrix} - \hat{\jmath}\begin{vmatrix}-2 & 0\\ 0 & 0\end{vmatrix} + \hat{k}\begin{vmatrix}-2 & 0\\ 0 & -1.5\end{vmatrix}$$

$\hat{\imath}(0 \cdot 0 - (-1.5) \cdot 0) = 0\hat{\imath}$

$-\hat{\jmath}((-2) \cdot 0 - 0 \cdot 0) = 0\hat{\jmath}$

$\hat{k}((-2)(-1.5) - 0 \cdot 0) = 3\hat{k}$

$$\implies \vec{v} \times \vec{w} = = \begin{bmatrix}0\\0\\3\end{bmatrix}$$



Now to verify: $\|\vec{v} \times \vec{w}\| = \sqrt{0^2 + 0^2 + 3^2} = \sqrt{9} = 3.$

cross product: generally

Let

$$\vec{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}, \qquad \vec{w} = \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}.$$

Then the cross product can be written as. $\vec{v} \times \vec{w} = \begin{vmatrix}\hat{\imath} & v_1 & w_1\\ \hat{\jmath} & v_2 & w_2\\ \hat{k} & v_3 & w_3\end{vmatrix}$

Expand along the first column

$$\vec{v} \times \vec{w} = \hat{\imath}\begin{vmatrix}v_2 & w_2\\ v_3 & w_3\end{vmatrix} - \hat{\jmath}\begin{vmatrix}v_1 & w_1\\ v_3 & w_3\end{vmatrix} + \hat{k}\begin{vmatrix}v_1 & w_1\\ v_2 & w_2\end{vmatrix}$$

cross product: generally

Compute determinants

$$\vec{v} \times \vec{w} = \hat{\imath}(v_2w_3 - v_3w_2) - \hat{\jmath}(v_1w_3 - v_3w_1) + \hat{k}(v_1w_2 - v_2w_1)\,.$$

Final component form $\vec{v} \times \vec{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}.$

properties of the determinant

Let A and B be $n \times n$ matrices and let k be a scalar

- $\det(kA) = k^n \cdot \det(A)$
- $\det(A^T) = \det(A)$
- $\det(AB) = \det(A)\det(B)$
- If A is invertible then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

- A matrix A is invertible if and only if $\det(A) \neq 0$
- A square matrix that has $\det(A) = 0$ is called singular and is not invertible

more on this next week....

matrix arithmetic: addition and subtraction

Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ where each matrix defines a function: $A(\vec{v}),\ B(\vec{v})$

Matrix Addition

"Apply both transformations to the same vector and add the results"

$$(A + B)(\vec{v}) \;=\; A(\vec{v}) + B(\vec{v})$$

For every input vector $\vec{v}$, you add the two output vectors

example

$A(\vec{v}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \vec{v}$ (horizontal push) $\quad B(\vec{v}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{v}$ (vertical push) $\quad$ Let $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

Then $A(\vec{v}) = \begin{pmatrix} x \\ 0 \end{pmatrix}, \quad B(\vec{v}) = \begin{pmatrix} 0 \\ y \end{pmatrix}$

Add them $(A + B)(\vec{v}) = \begin{pmatrix} x \\ y \end{pmatrix}$

Note: $A + B = I$ (the identity)

matrix arithmetic: addition and subtraction

example

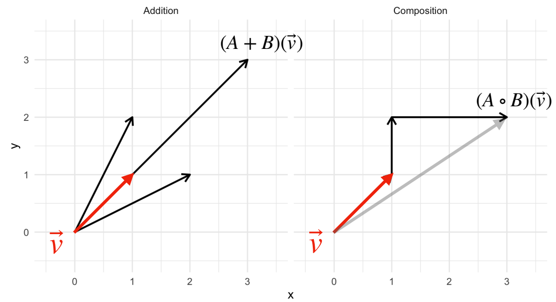
If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$: shear to the right

$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$: vertical stretch

Then:

$A \circ B$:up then right

$A + B$:right + up at the same time



matrix arithmetic: addition and subtraction

Let $A, B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where each matrix defines a function: $A(\vec{v}),\ B(\vec{v})$

Matrix Subtraction

"How much does A move $\vec{v}$ compared to B?"

$$(A - B)(\vec{v}) \ = \ A(\vec{v}) - B(\vec{v})$$

example