System of Equations & All Things 'Eigen'

Lecture 12

Termeh Shafie

linear system of equations

matrices let's us solve systems of equations

x	$6x - 3y + 2z = 7$
y	$x + 2y + 5z = 0$
z	$2x - 8y - z = -2$
unknown variables	list of equations relating the unknown variables

$$\begin{aligned} 6x - 3y + 2z &= 7 \\ x + 2y + 5z &= 0 \\ 2x - 8y - z &= -2 \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} 6 & -3 & 2 \\ 1 & 2 & 5 \\ 2 & -8 & -1 \end{bmatrix}}_{\text{coefficients}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{variables}} = \underbrace{\begin{bmatrix} 7 \\ 0 \\ 2 \end{bmatrix}}_{\substack{\text{matrix vector product} \\ \text{(constants)}}}$$

linear system of equations

matrices let's us solve systems of equations

$A\vec{x} = \vec{v}$

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linear system of equations

$A\vec{x} = \vec{v}$

linear system of equations

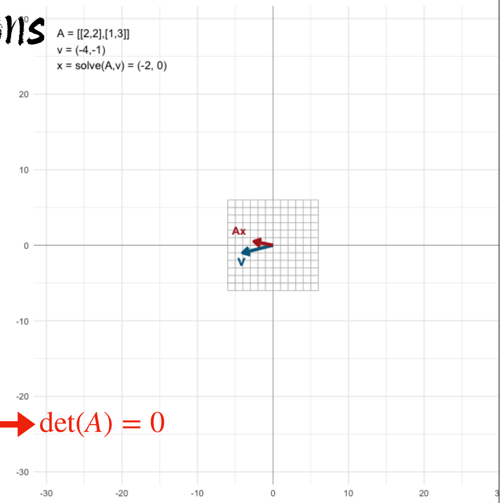
$A\vec{x} = \vec{v}$

example

$2x + 2y = -4$
$1x + 3y = -1$

$$\underbrace{\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -4 \\ -1 \end{bmatrix}}_{\vec{v}}$$

solving this depends on
- whether we squish the space into a smaller dimension? → $\det(A) = 0$
- or stay in 2D? → $\det(A) \neq 0$



5

linear system of equations

$A\vec{x} = \vec{v}$

example

$2x + 2y = -4$
$1x + 3y = -1$

$$\underbrace{\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -4 \\ -1 \end{bmatrix}}_{\vec{v}}$$

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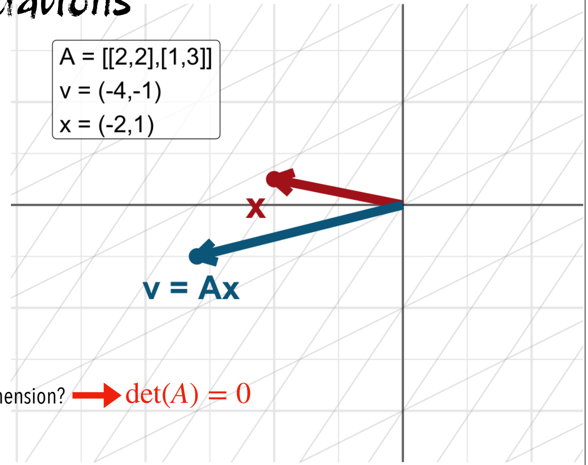


6

linear system of equations

If determinant is non-zero, there will be only solution landing on $\vec{v}$

this can be found by the inverse transformation which is represented by A^{-1}

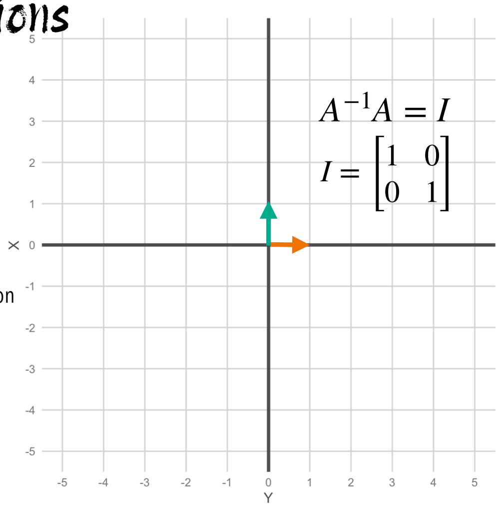
example

A corresponds to a $90°$ counterclockwise transformation

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

A^{-1} corresponds to a $90°$ clockwise transformation

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



$$A^{-1}A = I$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7

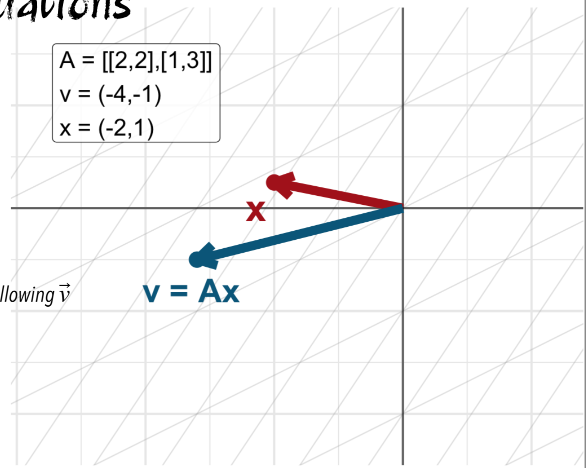
linear system of equations

$A\vec{x} = \vec{v}$

$A^{-1}A\vec{x} = A^{-1}\vec{v}$

$\vec{x} = A^{-1}\vec{v}$

you're playing the transformation in reverse and following $\vec{v}$



8

linear system of equations

$A\vec{x} = \vec{v}$

example

also applies to higher dimension…

$$\begin{aligned} 2x + 5y + 3z &= -3, \\ 4x + 0y + 8z &= 0, \\ 1x + 3y + 0z &= 2. \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}.$$

as long as A^{-1} exists, that is $\det(A) \neq 0$, then there is a solution

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when for example your transformations squishes everything into a line,
then there might still be a solution,
but you have to be “lucky” that the output vector lies on that line
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9

rank

“Rank” $\longleftrightarrow$ number of dimensions in the output

The rank of a matrix is defined as the number of linearly independent columns (or rows) of a matrix. If all of the columns are independent, we say that the matrix is of full rank. We denote the rank of matrix A by as rank(A).

- Full rank $\Longleftrightarrow$ nonsingular $\Longleftrightarrow$ invertible. All of these imply that A^{-1} exists.
- If a matrix is not of full rank, it is not invertible; i.e., it is singular.

example

Let $A = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$. The second column is twice the first column so the rank of the matrix is 1.

It is not full rank.

10

column space

“Rank” $\longleftrightarrow$ number of dimensions in the output

set of all possible outputs $A\vec{v}$ $\longleftrightarrow$ column space of the matrix

$$A = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$$

where $\hat{\imath}$ lands   where $\hat{\jmath}$ lands

the **column space** is the **span** of the columns of your matrix
so **rank** is number of dimensions in your column space

11

inverse of a matrix

Let A be an $n \times n$ matrix. It is invertible if and only if one can find a second $n \times n$ matrix, X, such that the product AX and the product XA both produce the $n \times n$ the identity matrix $I_{n \times n}$.

X is then the inverse of A, denoted by A^{-1} $\Longrightarrow$ $A \cdot A^{-1} = A^{-1} \cdot A = I$

Let A be an 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. A is invertible if and only if $ad - bc \neq 0$, If it is invertible then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The general rule: $A^{-1} = \dfrac{1}{|A|} C^T$ where C^T is the transpose of the matrix of cofactors of A.

Each element of C is the cofactor of the corresponding element of A.

12

system of linear equations

- Consider following system:

$$2x - y + 3z = 9$$
$$x + 4y - 5z = -6$$
$$x - y + z = 2$$

- This system of equations can be represented in matrix form as follows:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -5 \\ 1 & -1 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix} \implies A\vec{x} = \vec{v}$$

- Techniques for solving:
  - substitution
  - elimination
  - matrix inversion
  - Cramer's rule

we use this example in the following to illustrate each

substitution

- Choose the easiest variable to solve for and plug this expression for the variable into the other two equations you did not yet use
- Three things can happen:

1. There is the same number of equations as unknowns $\implies$ uniquely determined
2. There are more unknowns than equations i.e. infinite number of solutions $\implies$ underdetermined
3. There are more equations than unknowns (equations are contradictory) $\implies$ overdetermined

substitution

$$2x - y + 3z = 9 \quad (1)$$
$$x + 4y - 5z = -6 \quad (2)$$
$$x - y + z = 2 \quad (3)$$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -5 \\ 1 & -1 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix} \implies A\vec{x} = \vec{v}$$

- We solve equation (3) and substitute it into equation (2) and (1): $x = y - z + 2$

$\implies 2(y - z + 2) - y + 3z = 9 \implies 2y - 2z + 4 - y + 3z = 9 \implies y + z = 5 \quad (4)$

$\implies (y - z + 2) + 4y - 5z = -6 \implies y - z + 2 + 4y - 5z = -6 \implies 5y - 6z = -8 \quad (5)$

- Now solve equations (4) and (5):

From (4) we have $y = 5 - z$ which is substituted into equation (5)

$5(5 - z) - 6z = -8 \implies 25 - 5z - 6z = -8 \implies -11z = -33 \implies z = 3$

Substitute back into y: $y = 5 - 3 = 2$

and finally from $x = y - z + 2$ we get $x = 2 - 3 + 2 = 1$

Gaussian elimination

- We eliminate one variable by combining equations

$$2x - y + 3z = 9 \quad (1)$$
$$x + 4y - 5z = -6 \quad (2)$$
$$x - y + z = 2 \quad (3)$$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -5 \\ 1 & -1 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix} \implies A\vec{x} = \vec{v}$$

- Eliminate x:
  - subtracting equation (3) from (2): $(x + 4y - 5z) - (x - y + z) = -6 - 2 \implies 5y - 6z = -8 \quad (4)$
  - subtract 2 × equation (3) from equation (1): $(2x - y + 3z) - 2(x - y + z) = 9 - 2(2) \implies y + z = 5 \quad (5)$
- Solve for y and z:
  - from equation (5) we get a new equation: $y = 5 - z \quad (6)$ which is substituted into (4):

  $5(5 - z) - 6z = -8 \implies 25 - 5z - 6z = -8 \implies -11z = -33 \implies z = 3$

  - from (6) we get $y = 5 - 3 = 2$ and from (3) we get $x = y - z + 2 \implies x = 2 - 3 + 2 = 1$

matrix inversion

$$2x - y + 3z = 9 \quad (1)$$
$$x + 4y - 5z = -6 \quad (2)$$
$$x - y + z = 2 \quad (3)$$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -5 \\ 1 & -1 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix} \implies A\vec{x} = \vec{v}$$

- Find the determinant of A to make sure it's invertible (via cofactor expansion along first row): $\det(A) = -11$
- This is not zero, the matrix is nonsingular, and we can invert it
- To invert the matrix we compute 9 minors:

$M_{11} = -1,\ M_{12} = 6,\ M_{13} = -5,\ M_{21} = 2,\ M_{22} = -1,\ M_{23} = -1,\ M_{31} = -7,\ M_{32} = -13,\ M_{33} = 9$

- Now use the formula for the inverse from the previous lecture $A^{-1} = \frac{1}{|A|}C^T$ where $C_{ij} = (-1)^{i+j}M_{i,j}$
- This gives: $\mathbf{A}^{-1} = \frac{1}{-11} \cdot \begin{bmatrix} -1 & -2 & -7 \\ -6 & -1 & 13 \\ -5 & 1 & 9 \end{bmatrix}$
- And finally we multiply with $\vec{v}$ to get the solution: $\vec{x} = A^{-1}\vec{v} = \frac{1}{-11} \cdot \begin{bmatrix} -1 & -2 & -7 \\ -6 & -1 & 13 \\ -5 & 1 & 9 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

17

Cramer's rule

- only works when there an equal number of equations and unknowns (i.e. a square matrix A) and A is nonsingular
- this rules states that we can solve for $\vec{x}$ using the formula:

$$x_i = \frac{|B_i|}{|A|}$$

where the matrix B_i is formed by replacing the ith column of A (the column corresponding to variable x_i) with $\vec{v}$

1. take the determinant of A, to check to make sure that we can apply this rule and determine the denominator of each x_i
2. we form the B_i by replacing each of the three columns by $\vec{v}$
3. compute the determinants of the matrices B_i
4. apply Cramer's rule according to formula

18

Cramer's rule

$$2x - y + 3z = 9 \quad (1)$$
$$x + 4y - 5z = -6 \quad (2)$$
$$x - y + z = 2 \quad (3)$$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -5 \\ 1 & -1 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix} \implies A\vec{x} = \vec{v}$$

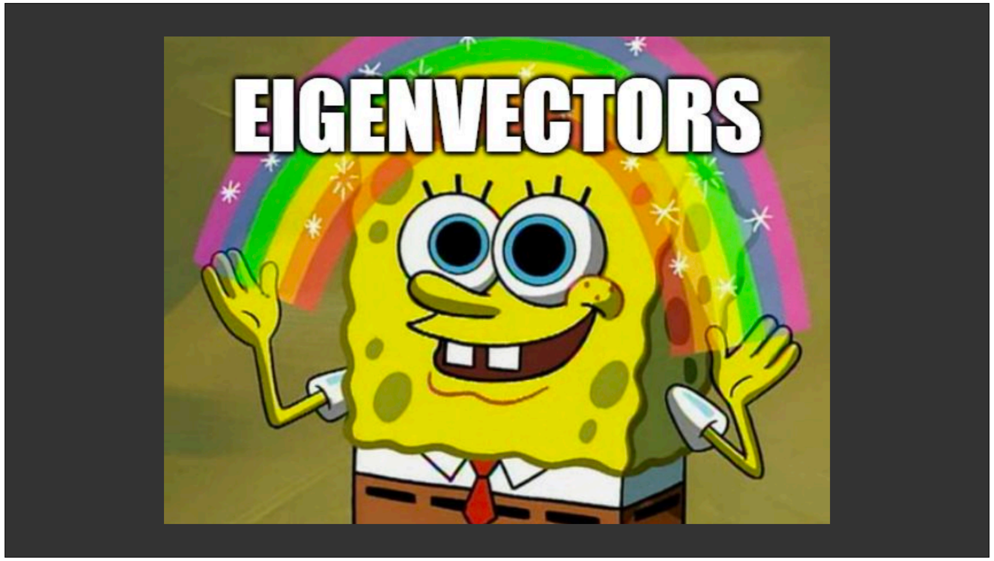
- we already know the determinant of A: $\det(A) = -11$
- form the B_i by replacing each of the three columns by $\vec{v}$

$$B_1 = \begin{bmatrix} 9 & -1 & 3 \\ -6 & 4 & -5 \\ 2 & -1 & 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 2 & 9 & 3 \\ 1 & -6 & -5 \\ 1 & 2 & 1 \end{bmatrix},\ B_3 = \begin{bmatrix} 2 & -1 & 9 \\ 1 & 4 & -6 \\ 1 & -1 & 2 \end{bmatrix}$$

- compute determinant for each B_i: $\det(B_1) = -11,\ \det(B_2) = -22,\ \det(B_3) = -33$
- apply Cramer's rule:

$$x = \frac{\det(B_1)}{\det(A)},\quad y = \frac{\det(B_2)}{\det(A)},\quad z = \frac{\det(B_3)}{\det(A)} \implies x = \frac{-11}{-11} = 1,\quad y = \frac{-22}{-11} = 2,\quad z = \frac{-33}{-11} = 3$$

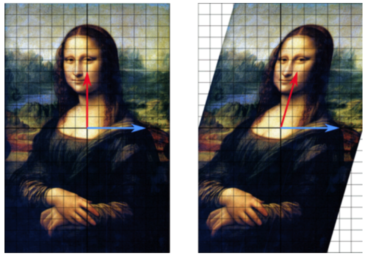
19



20

linear transformations

A linear transformation is an operation that
stretches, squishes, rotates, or otherwise transforms a space



source: https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

Think of scaling an image, rotating it, or stretching it in one direction.
Eigenvectors and eigenvalues reveal the fundamental "axes" of a transformation.

let's recap some important definitions

- A set of vectors $\in \mathbb{R}^n$ is called a basis, if they are linearly independent and every vector $\in \mathbb{R}^n$ can be expressed as a linear combination of these vectors.

- A set of n vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ is linearly independent if no vector in the set can be expressed as a linear combination of the remaining $n-1$ vectors. In other words, the only solution to $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = 0$ is $c_1 = c_2 = \ldots = c_n = 0$ (where c_i are scalars)

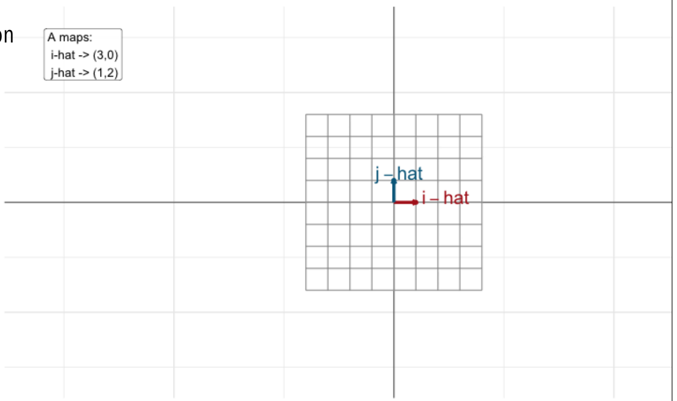
- In other words, the coefficients $c_1, c_2, \ldots, c_n$ must all be zero for the linear combination to result in the zero vector and properly characterizing linear independence.

eigenvectors

example

consider a transformation based on

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$



eigenvectors

example

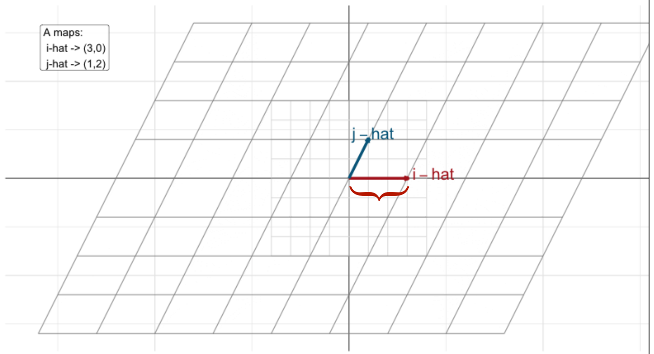
consider a transformation based on

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

most vectors get knocked of their span after the transformation

but some stay put and only get stretched/squished/reversed

$\hat{\imath}$ is such a vector:
it moves on its span but stretched by a factor of 3



eigenvectors

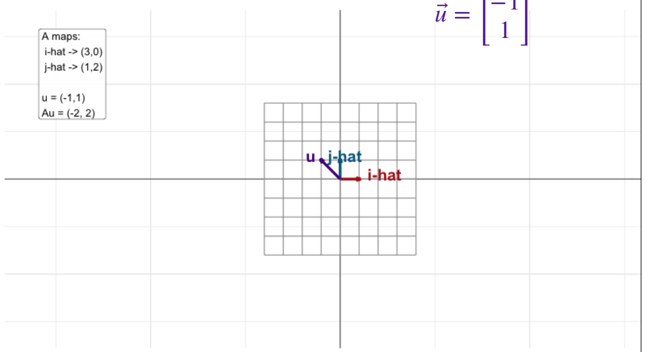
example

consider a transformation based on

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

here's another vector that remains on its own span

$$\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



eigenvectors

example

consider a transformation based on

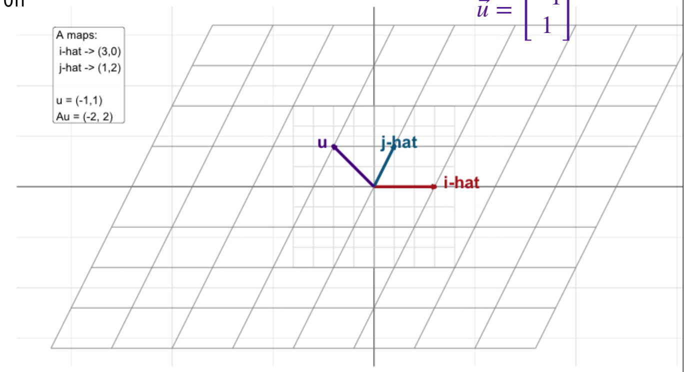
$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

gets stretched by a factor of 2

here's another vector that remains on its own span

$$\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



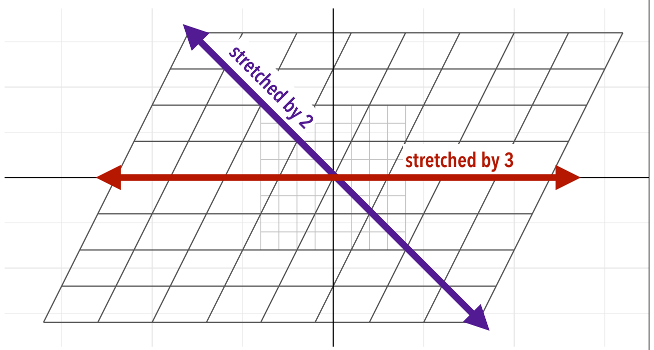
eigenvectors

example

consider a transformation based on

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

for this transformation these two vectors are the only ones staying on their span



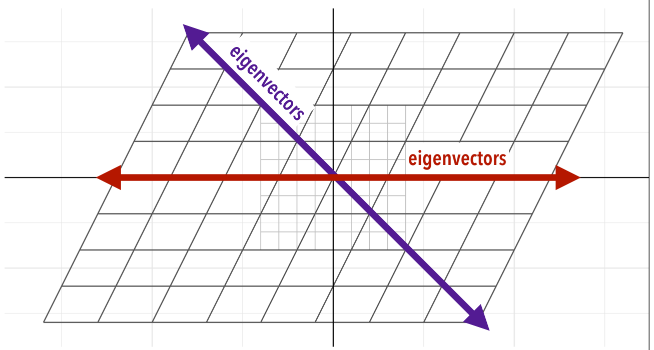
eigenvectors

example

consider a transformation based on

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

these are the **eigenvectors** of this transformation



eigenvectors

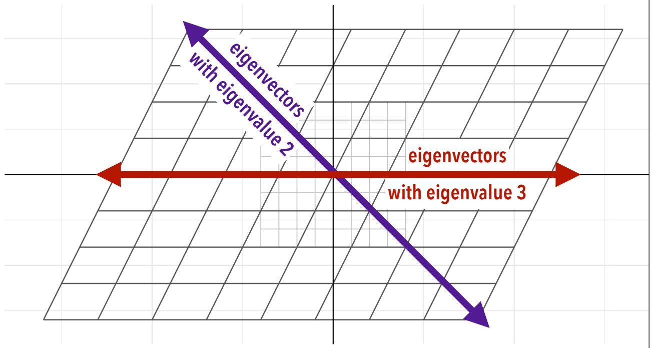
example

consider a transformation based on

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

these are the **eigenvectors** of this transformation

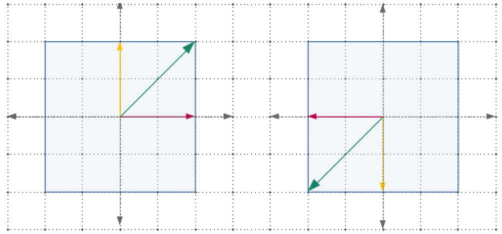
the factor by which they are stretched/squished are the **eigenvalues**



eigenvectors and eigenvalues

example

In 180 degree rotation of square, all vectors are still laying on the same span, but their direction is reversed. Hence, all vectors are eigenvectors, having an eigenvalue of -1.



Note: In case of 3d rotation transformation of cube, the eigenvector gives the axis of rotation.

eigenvectors and eigenvalues

transformation matrix

$$A\vec{v} = \lambda\vec{v}$$

eigenvectors

eigenvalue

eigenvectors

we need to solve for $\vec{v}$ and λ

but this is matrix vector product on left hand side and scalar vector product right hand side…. ☹️

eigenvectors and eigenvalues

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = (\lambda I)\vec{v}$$

$$A\vec{v} - (\lambda I)\vec{v} = \vec{0} \implies \boxed{(A - \lambda I)\vec{v} = \vec{0}}$$

create a matrix which has the effect of scaling any vector by a factor of λ:

$$\begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \lambda I$$

eigenvectors and eigenvalues

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = (\lambda I)\vec{v}$$

we want a non-zero solution

$$A\vec{v} - (\lambda I)\vec{v} = \vec{0} \implies \underbrace{(A - \lambda I)}\vec{v} = \vec{0}$$

example:
$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 1 & 5-\lambda & 9 \\ 2 & 6 & 5-\lambda \end{bmatrix}$$

33

eigenvectors and eigenvalues

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = (\lambda I)\vec{v}$$

$$A\vec{v} - (\lambda I)\vec{v} = \vec{0} \implies (A - \lambda I)\vec{v} = \vec{0}$$

can only happen when $\det(A - \lambda I) = 0$

so we need to find the λ that makes this determinant 0

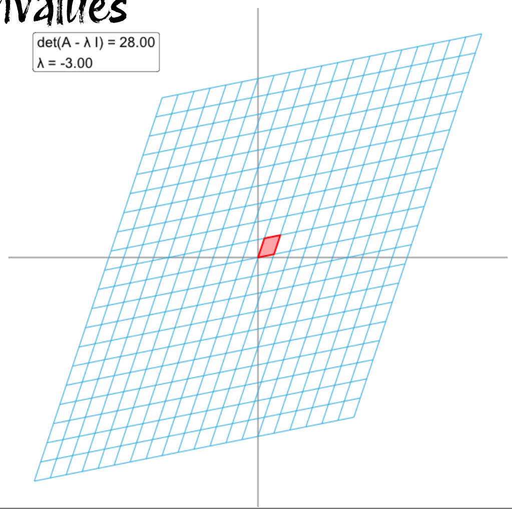
34

eigenvectors and eigenvalues

example

consider a transformation based on

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$



35

eigenvectors and eigenvalues

example

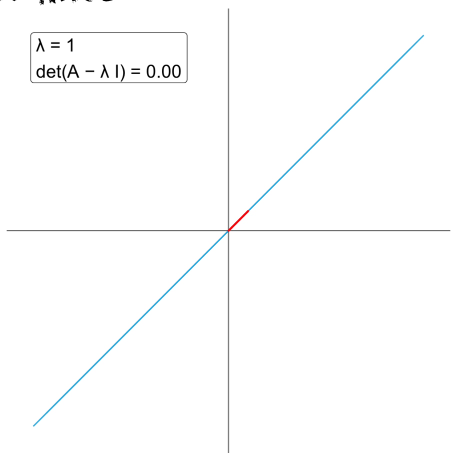
consider a transformation based on

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2-1.00 & 2 \\ 1 & 3-1.00 \end{bmatrix}\right) = 0.00$$

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\vec{v} = 1\,\vec{v}$$



36

eigenvectors and eigenvalues

example

consider a transformation based on

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

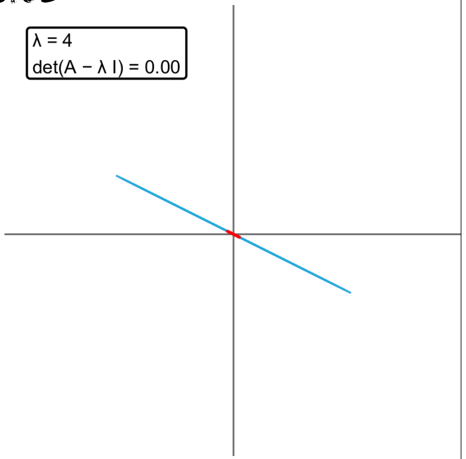
$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0$$

$$(2-\lambda)(3-\lambda) - (2)(1) = 0$$

$$(\lambda - 4)(\lambda - 1) = 0$$

$$\lambda = 1,\ \lambda = 4$$



eigenvectors and eigenvalues

back to previous example

consider a transformation based on

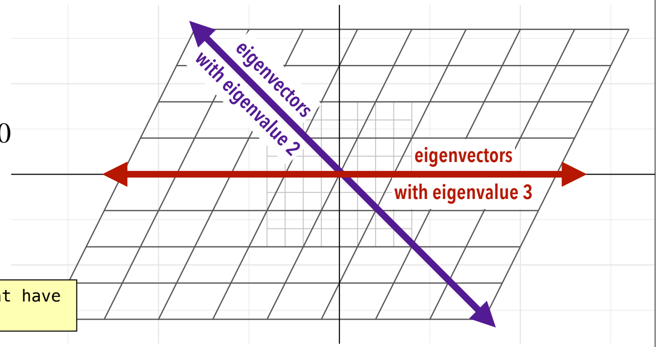
$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(3-\lambda)(2-\lambda) = 0$$

$$\lambda = 2,\ \lambda = 3$$

what are the eigenvectors that have these eigenvalues?



eigenvectors and eigenvalues

back to previous example

consider a transformation based on

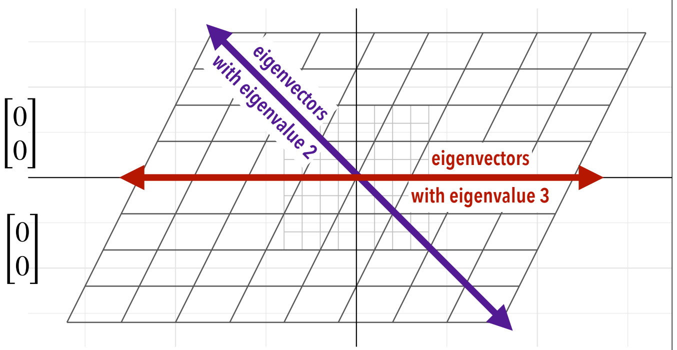
$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for $\lambda = 2$:

$$\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



eigenvectors and eigenvalues

some notes

➡ not all 2D transformations have eigenvectors

➡ one eigenvalue can have more than a line of eigenvectors

for example:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

where the only eigenvalues is 2 but every vector in the plane gets to be an eigenvector

eigenvectors and eigenvalues

Let A be a $n \times n$ matrix.

1. An eigenvector of A is a nonzero vector $\vec{v}$ in $\mathbb{R}^n$ such that $A\vec{v} = \lambda\vec{v}$, for some scalar λ
2. An eigenvalue of A is a scalar λ such that the equation $A\vec{v} = \lambda\vec{v}$ has a *non-trivial** solution

If $A\vec{v} = \lambda\vec{v}$ for $\vec{v} \neq 0$, we say that λ is the eigenvalue for $\vec{v}$, and that $\vec{v}$ is an eigenvector for λ.

*means that the solution vector $\vec{v}$ is not the zero vector ($\vec{v} \neq \vec{0}$), and ensures that it represents a meaningful direction in the vector space.

verifying eigenvectors

How to check if a given $\vec{v}$ is the eigenvector of a given matrix A

- multiply $\vec{v}$ by A and see if $A\vec{v}$ is a scalar multiple of $\vec{v}$, i.e. $A\vec{v} = \lambda\vec{v}$
- what happens when a matrix hits a vector?

example

Consider matrix $A = \begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix}$ and vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

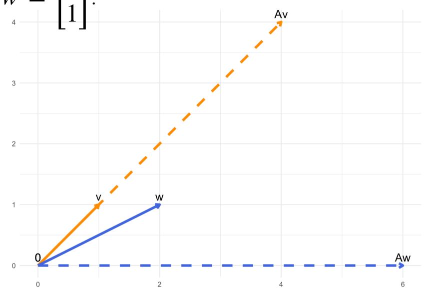
Which are eigenvectors? What are their eigenvalues?

$$A\vec{v} = \begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\vec{v}$$

$\implies \vec{v}$ is an eigenvector of A

$$A\vec{w} = \begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$\implies \vec{w}$ is not an eigenvector of A



the characteristic polynomial

Let A be a $n \times n$ matrix. The characteristic polynomial of A is the function $f(\lambda)$ given by

$$f(\lambda) = \det(A - \lambda I)$$

- The characteristic polynomial is in fact a polynomial
- The point of the characteristic polynomial is that we can use it to compute eigenvalues
- Finding the characteristic polynomial means computing the determinant of the matrix $\det(A - \lambda I)$ whose entries contain the unknown λ

Eigenvalues are roots of the characteristic polynomial

Let A be a $n \times n$ matrix and let $f(\lambda) = \det(A - \lambda I)$ be its characteristic polynomial.

Then a number λ_0 is an eigenvalue of A if and only if $f(\lambda_0) = 0$.

trace

Let A be an $n \times n$ matrix. The trace of A, denoted $tr(A)$, is the sum of the diagonal elements of A. That is,

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

Properties of trace:

- $tr(A + B) = tr(A) + tr(B)$
- $tr(A - B) = tr(A) - tr(B)$
- $tr(kA) = k \cdot tr(A)$
- $tr(AB) = tr(BA)$
- $tr(A^T) = tr(A)$

the characteristic polynomial: a shortcut

The characteristic polynomial for a 2×2 matrix

Let A be a 2×2 matrix. All coefficients of the characteristic polynomial can be found via

$$f(\lambda) = \lambda^2 + tr(A)\lambda + \det(A)$$

this is generally the fastest way to compute the characteristic polynomial of a 2×2 matrix.

the process so far...

1. **Find Eigenvalues:**
   Determine the eigenvalues λ of the matrix A by solving the characteristic polynomial $\det(A - \lambda I) = 0$ where I is the identity matrix.
2. **Find Eigenvectors:**
   For each eigenvalue, find the corresponding eigenvectors by solving $(A - \lambda I)\vec{v} = 0$ which is equivalent to solving $A\vec{v} = \lambda\vec{v}$

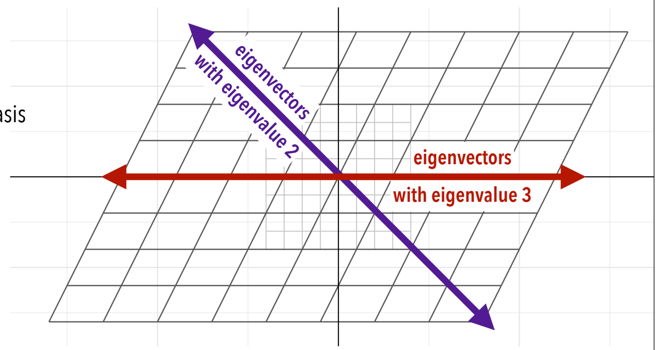
eigen basis

back to previous example

consider a transformation based on

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

let's now use eigenvectors as basis



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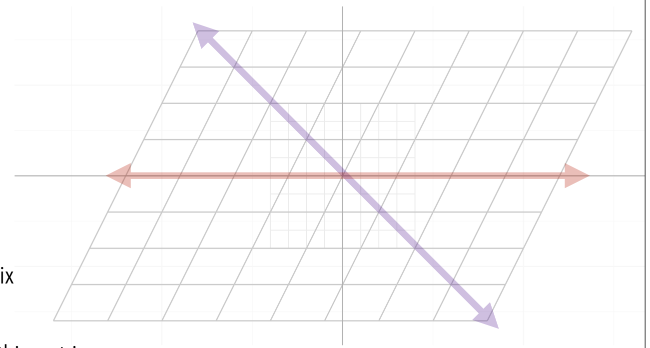

let's now use eigenvectors as basis
by putting them as columns in a matrix
known as **the change of basis matrix**

then sandwich the original matrix by this matrix

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

diagonal matrix with eigenvalues in diagonal

the result is the same transformation but from the perspective of the new basis vectors coordinate system

this is called the **eigenbasis**

diagonalization

- Diagonal matrices are the easiest kind of matrices to understand: they just scale the coordinate directions by their diagonal entries.
- Matrix diagonalization is powerful: it transforms a given square matrix into a diagonal matrix,
  - which is much easier to analyze and compute because their non-diagonal elements are zero
  - e.g. calculations like powers and determinants easy to perform

An $n \times n$ matrix A is diagonalizable if it is similar to a diagonal matrix: that is, if there exists an invertible$n \times n$ matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$

powers of diagonalizable matrices

Multiplying diagonal matrices together just multiplies their diagonal entries:

$$\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} \begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix} = \begin{bmatrix} x_1y_1 & 0 & 0 \\ 0 & x_2y_2 & 0 \\ 0 & 0 & x_3y_3 \end{bmatrix}$$

so it is easy to take powers of a diagonal matrix:

$$\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}^n = \begin{bmatrix} x^n & 0 & 0 \\ 0 & y^n & 0 \\ 0 & 0 & z^n \end{bmatrix}$$

$\Longrightarrow$ if $A = QDQ^{-1}$ where D is the diagonal matrix, then $A^n = QD^nQ^{-1}$

diagonalization theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

simply put: a matrix is diagonalizable if it has distinct eigenvalues or, if it has repeated eigenvalues, it still has enough independent eigenvectors to match its dimensionality

- the eigenvalues determine the entries of the diagonal matrix
- the eigenvectors form the columns of a matrix Q
- the transformation reflects how the original matrix A can be simplified, highlighting the intrinsic properties of A

$$A = QDQ^{-1} \quad \text{where} \quad Q = \begin{bmatrix} | & | & | & | \\ \vec{v_1} & \vec{v_2} & \dots & \vec{v_1} \\ | & | & | & | \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the corresponding eigenvalues

the process continued...

$A = QDQ^{-1}$

1. Find Eigenvalues:
   Determine the eigenvalues λ of the matrix A by solving the characteristic polynomial $\det(A - \lambda I) = 0$ where I is the identity matrix.
2. Find Eigenvectors:
   For each eigenvalue, find the corresponding eigenvectors by solving $(A - \lambda I)\vec{v} = 0$
3. Form Matrices:
   Arrange the eigenvectors as columns in a matrix Q and the eigenvalues along the diagonal in a matrix D.
4. Construct the Diagonalization:
   If A is diagonalizable you can express it as $A = QDQ^{-1}$

the recipe

$A = QDQ^{-1}$

Let A be an $n \times n$ matrix representing a transformation. To diagonalize A:

1. Find the eigenvalues of A using the characteristic polynomial.
2. For each eigenvalue λ of A, compute the basis B_λ for the $\lambda-$eigenspace.
3. If there are fewer than n total vectors in all of the eigenspace bases B_λ, then the matrix is not diagonalizable.
4. Otherwise, the n vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ in the eigenspace bases are linearly independent, and

$$A = QDQ^{-1} \text{ for } Q = \begin{bmatrix} | & | & | & | \\ \vec{v_1} & \vec{v_2} & \ldots & \vec{v_1} \\ | & | & | & | \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

where λ_i is the eigenvalue for $\vec{v_i}$.

eigendecomposition summarized

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be the eigenvectors of matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be corresponding eigenvalues

Consider now a matrix Q whose columns are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

We have now

$$AQ = A \begin{bmatrix} | & | & | & | \\ \vec{v_1} & \vec{v_2} & \ldots & \vec{v_1} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ A\vec{v_1} & A\vec{v_2} & \ldots & A\vec{v_1} \\ | & | & | & | \end{bmatrix}$$

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$$= \begin{bmatrix} | & | & | & | \\ \lambda_1\vec{v_1} & \lambda_2\vec{v_2} & \ldots & \lambda_n\vec{v_1} \\ | & | & | & | \end{bmatrix}$$

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$$= \begin{bmatrix} | & | & | & | \\ \vec{v_1} & \vec{v_2} & \ldots & \vec{v_1} \\ | & | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \qquad = QD$$

eigendecomposition summarized

- If Q^{-1} exists, then we can write
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    2. $Q^{-1}AQ = D$ diagonalization of A

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    - If the columns of Q are linearly independent
    - i.e. if A has n linearly independent eigenvectors
    - i.e. if A has n distinct eigenvalues

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    1. $A = QDQ^{-1}$ eigenvalue decomposition
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    - i.e. if A has n distinct eigenvalues
- If A is symmetric, we get an even more convenient situation
    - The eigenvalues are orthogonal